Lipschitz Functions

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**Definition 1** Let \( f(x) \) be defined on an interval \( I \) and suppose we can find two positive constants \( M \) and \( \alpha \) such that

\[
|f(x_1) - f(x_2)| \leq M|x_1 - x_2|^{\alpha}
\]

for all \( x_1, x_2 \in I \).

Then \( f \) is said to satisfy a Lipschitz Condition of order \( \alpha \) and we say that \( f \in \text{Lip}(\alpha) \).

**Example 1** Take \( f(x) = x \) on the interval \([a, b]\). Then

\[
|f(x_1) - f(x_2)| = |x_1 - x_2|
\]

That implies that \( f \in \text{Lip}(1) \).

Now take \( f(x) = x^2 \) on the interval \([a, b]\). Then

\[
|f(x_1) - f(x_2)| = |x_1^2 - x_2^2| = |x_1 - x_2||x_1 + x_2| \leq M|x_1 - x_2|
\]

with \( M = 2\max(|a|, |b|) \). Hence, again \( f \in \text{Lip}(1) \).

The function \( f(x) = 1/x \) on \((0, 1)\). Is it \( \text{Lip}(1) \)? How about \( \text{Lip}(1/2) \)? How about \( \text{Lip}(\alpha) \)?

**Theorem 1** \( \text{Lip}(\alpha) \) is a linear space.

**Proof** We will look at a part of this proof. Let \( f, g \in \text{Lip}(\alpha) \).

\[
(f + g)(x) \in \text{Lip}(\alpha)
\]

Then,

\[
|f(x) + g(x) - f(y) + g(y)| \leq M|x - y|^{\alpha}
\]

If \( f \in \text{Lip}(\alpha) \) it implies that

\[
|f(x) - f(y)| \leq M_1|x - y|^{\alpha}
\]

If \( g \in \text{Lip}(\alpha) \) it implies that
\[ |g(x) - g(y)| \leq M_2|x - y|^{\alpha} \]

If \((f + g) \in \text{Lip}(\alpha)\) it implies that

\[ |(f + g)(x) - (f + g)(y)| \leq M_3|x - y|^{\alpha} \]

\[ |f(x) + g(x) - f(y) - g(y)| = \]

\[ |f(x) - f(y) + g(x) - g(y)| = \]

\[ |f(x) - f(y)| + |g(x) - g(y)| = \]

By the triangle inequality,

\[ |f(x) - f(y)| + |g(x) - g(y)| \leq M_1|x - y|^{\alpha} + M_2|x - y|^{\alpha} = |M_1 + M_2||x - y|^{\alpha} \]

**Theorem 2** If \(f \in \text{Lip}(\alpha)\) with \(\alpha > 1\) then \(f = \text{constant}\).

**Proof** Left as homework for everyone.

1 Lipschitz and Continuity

**Theorem 3** If \(f \in \text{Lip}(\alpha)\) on \(I\), then \(f\) is continuous; indeed, uniformly continuous on \(I\).

Last time we did continuity with \(\epsilon\) and \(\delta\). An alternative definition of continuity familiar from calculus is: \(f\) is continuous at \(x = c\) if:

- \(f(c)\) exists
- \(\lim_{x \to c} f(x)\) exists
- \(\lim_{x \to c} f(x) = f(c)\)

In order to be continuous, if \(|x - x_0| < \delta\), then \(|f(x) - f(x_0)| < \epsilon\).

**Proof**

\[ |f(x) - f(c)| \leq M|x - c|^{\alpha} \]

\[ \lim_{x \to c}|f(x) - f(c)| \leq M \lim_{x \to c}|x - c|^{\alpha} = 0 \]

This implies

\[ \lim_{x \to c} f(x) = f(c) \]

How about continuous implies \(\text{Lip}(\alpha)\)?
2 Lipschitz and Differentiability

**Theorem 4** If \( f \in \text{Lip}(\alpha) \), it may fail to be differentiable, but if it possesses a derivative satisfying \( |f'(x)| \leq M \) then \( f \in \text{Lip}(1) \).

In order to be differentiable,
\[
\lim_{x \to \infty} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)
\]

By the Mean Value Theorem,
\[
\Rightarrow \frac{|f(x) - f(y)|}{|x - y|} = |f'(c)|, c \in (x, y)
\]

This implies that
\[
|f(x) - f(y)| = |f'(c)||x - y|
\]

Now if
\[
|f'(x)|
\]
exists and is bounded by \( M \), then
\[
|f(x) - f(y)| \leq M|x - y|
\]

which implies \( f \in \text{Lip}(1) \).