1 Introduction

I will discuss the section of infinitely differentiable functions, their properties, and some useful examples to show the difference between functions that are infinitely differentiable, and those that aren’t.

2 Differentiable functions

Definition: A real function is said to be differentiable at a point if its derivative exists at that point. The notion of differentiability can also be extended to complex functions (leading to the Cauchy-Riemann equations and the theory of holomorphic functions).

3 Infinitely Differentiable Functions

Definition: If \( f(x) \in C^n \) on the interval \([a, b]\) for \( n = 0, 1, 2, \ldots \), then \( f \) is called Infinitely Differentiable on \([a, b]\). We shall write \( C^\infty [a, b] \) for the class of such functions.

Use the definition to find the Taylor series (centered at \( c \)) for the function

**Example 3.1** \( f(x) = e^x, \text{ where } c = 0 \)
Example 3.2 $f(x) = e^{-2x}$

Example 3.3 $f(x) = \cos(x)$, where $c = \frac{\pi}{4}$

Example 3.4 $f(x) = \ln x$, where $c = 1$

Example 3.5 $f(x) = \frac{1}{1 + x^2}$ is $C^\infty$

4 Taylor Series

**Definition:** If a function $f$ has derivatives of all orders at $x=c$, then the series

$$\sum_{n=0}^{\infty} f^n(c) \frac{(x-c)^n}{n!}$$

is called the Taylor series. If $c = 0$ then the series is a Mclaurin Series for $f$

$$\lim_{n \to \infty} \frac{f^{(n+1)}(z)}{(n+1)!} (x-c)^{n+1} = 0$$

for every $x$ in $I$.

5 Summary of Taylor Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(c)}{n!} (x-c)^n$$

the coefficients $x_n$ are given by: $x_n = \frac{f^n(c)}{n!}$ where $f^n(c)$ is the $n$th derivative of $f$ evaluated at a point $c$.

The functions of class $C^\infty[a,b]$ form a linear space. If $f \in C^\infty[a,b]$ and $x_0 \in [a,b]$ we may form the Taylor expansion. The Taylor expansion for a given $x$, this series may or may not converge. If it converges, it may or may not converge to $f(x)$.

**Proof:**

Facts to prove:

Example 5.1 $f$ is continuous

Example 5.2 $f$ is differentiable and $f'(0)=0$
\[ f^n(0) = 0 \]

Let \( p(x), q(x) \in \mathbb{R} \) be polynomials and define

if \( f \in C^\infty \), then we can certainly write a taylor series for \( f \). However, it requires that this Taylor series actually converge (at least across some radius of convergence, such that the series converges absolutely for all real or complex numbers.)

\[
f(x) = \begin{cases} 
e^{-\frac{x^2}{2}} & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}
\]

Then \( f \in C^\infty \), and for any \( n \) greater than or equal to 0, \( f^{(n)}(0) = 0 \). So the Taylor series around 0 is 0; since \( f(x) > 0 \) for all \( x \) not equal to 0, clearly it does not converge to \( f \).

**Example 5.3** \( f \) is \( C^\infty(R) \) and \( f^n(0) = 0 \)

q.e.d.