MATH 3912 - Assignment 5

1. Suppose \( p \in P_2 \) and \( p(x_0) = p'(x_0) = p''(x_0) = 0 \). Show that \( p \) must be identically zero.

2. Suppose \( p \in P_2 \) and suppose that \( p(x_1) = p'(x_1) = 0 \) and \( p(x_2) = 0 \) for some fixed numbers \( x_1 \neq x_2 \). Show that \( p \) must then be identically zero.

3. Show that if \( p \in P_n \) and \( p \) has a root of total multiplicity \( n + 1 \) then \( p \) must be identically zero. Recall that \( p \) has a root of multiplicity \( n + 1 \) at \( x = a \) if \( p(a) = p'(a) = \ldots = p^{(n)}(a) = 0 \).

4. If \( p(x) \) is a polynomial of even degree, what is \( \lim_{x \to \infty} p(x) \) and \( \lim_{x \to -\infty} p(x) \)? Evaluate the same limits in case \( p \) is a polynomial of odd degree.

5. If \( f(x) \) is a polynomial then \( \lim_{n \to \infty} f^{(n)}(x) = 0 \) for all \( x \).

6. Show that \( f(x) = 2^x \) can coincide with a polynomial at only a finite number of points.

7. Suppose \( f \) is real analytic for all \( x \) and \( f^{(k)}(x) > 0 \) for \( k = 0, 1, \ldots \). Then \( f \) can not coincide with a polynomial infinitely often.

8. Suppose \( X = C^1([a, b]) \) and \( x_0 \) is a fixed point in \([a, b] \). Define \( L(f) = f'(x_0) \) for all \( f \in X \). Is \( L \) a linear functional?

9. Suppose \( X = C^0([a, b]) \). Define two functionals \( L \) and \( G \) via

\[
L(f) = \int_a^b x^2 f(x) \, dx \quad \text{and} \quad G(f) = \int_a^b (f(x))^2 \, dx
\]

Which of these functions is linear?

10. Suppose \( X = P^n \), \( x_0 \) is a fixed point in \([a, b] \), and \( f \) is a continous function defined on the interval \([a, b]\). Define \( L(p) = (f \circ p)(x_0) \). Is \( L \) a linear functional?

11. Suppose \( X = C^1([a, b]) \). Define \( L(f) = f' \) for all \( f \in X \). What is the range of this map \( L \)? Is the map \( L \) linear? Is \( L \) a linear functional?

12. Let \( X \) be the space of \( n \times n \) matrices and define \( L(A) = \det(A) \). Is \( L \) a linear functional?

13. Let \( X \) be the space of \( n \times n \) matrices. For a matrix \( A \in X \) define the trace of \( A \) as

\[
\text{trace}(A) = \text{trace} \begin{pmatrix}
a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\
a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n,1} & a_{n,2} & \ldots & a_{n,n}
\end{pmatrix} = a_{1,1} + a_{2,2} + \ldots + a_{n,n}
\]

Is \( L(A) = \text{trace}(A) \) a linear operator?