1. Suppose \( f(x) = x(x^2 - 1) \), defined on the interval \((-1, 1)\). Does Rolle’s theorem apply? If so, find the point \( c \in (-1, 1) \) that is guaranteed to exist (there could be more than one correct answer).

2. If \( f(x) = x(x^2 - 1) \) on \((-1, 1)\) then \( f(-1) = f(0) = f(1) = 0 \), so the Generalized Rolle’s theorem applies. Find all points that are guaranteed to exist by the generalized Rolle’s theorem in this case. \textit{Hint:} There should be two points where \( f' = 0 \) and one point where \( f'' = 0 \).

3. Show that \( f(x) = x^\frac{4}{3} \) is \( C^1(\mathbb{R}) \) but not \( C^2(\mathbb{R}) \). What about \( f(x) = x^\frac{5}{3} \)?

4. Find a function \( f(x) \) that is \( C^n \) but not \( C^{n+1} \) on the real line (you can not use the function below even though it does have the right properties, but you could make up a function that’s similar to the ones above).

5. Show that the function \( f \) defined below is \( C^{k-1}(\mathbb{R}) \) but not \( C^k(\mathbb{R}) \) on \((-1, 1)\).
   \[
   f(x) = \begin{cases} 
   x^k & \text{for } x \geq 0 \\
   0 & \text{for } x < 0 
   \end{cases}
   \]

6. Show that the function \( f \) defined below is continuous but not differentiable at \( x = 0 \) (which would put this function into \( C^k \))
   \[
   f(x) = \begin{cases} 
   x \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\
   0 & \text{for } x = 0 
   \end{cases}
   \]

7. Find the 6-th Taylor polynomial for the function \( f(x) = \cos(x) \) around \( x = 0 \). Then use that polynomial to approximate the value of \( \cos(0.5) \). Use a calculator to compare your approximation with the actual value of \( \cos(0.5) \) (don’t forget to switch your calculator into radian).

8. Find the 5-th Taylor polynomial for the function \( f(x) = \ln x \) around \( x = 1 \) and use it to approximate the value of \( \ln(1.5) \). Use a calculator to compare your approximation with the actual value.

9. Find the 3-rd Taylor polynomial for the function \( f(x) = x(x^2 - 1) \) around \( x = -1 \). Then find the 10-th and the 20-th Taylor polynomial for that function around \( x = -1 \).

10. Show that if \( f(x) \) is a polynomial then \( \lim_{x \to \infty} f^{(n)}(x) = 0 \) for all \( x \). Is the converse also true?

11. For many functions such as \( e^x \) and \( \sin(x) \) or \( \cos(x) \) we can use higher and higher Taylor polynomials to get better and better approximation (polynomials of higher and higher degrees). Does this process work for every function?