Panel 1

Review (and Correction)

If \( f(t,t) \) is a function, \( y(t) = \langle x(t), y(t) \rangle \) a curve

Then

\[
\begin{align*}
\int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2} \, dt & = \int_0^1 \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \, dt \\
\int_0^1 f(x(t), y(t)) \, dt & = \int_0^1 y(t) \, dt \\
\end{align*}
\]

If \( \mathbf{F} = \langle M, N \rangle \),

\[
\int_0^1 \mathbf{F} \cdot d\mathbf{r} = \int_0^1 M \, dx + N \, dy
\]

\( \mathbf{r} = \langle dx, dy \rangle \)

Panel 2

Let \( f(x,y) = x^2 - xy + y^2 \), \( F(x,y) = \langle 2x - y, 2y - x \rangle \), \( D = \{(x,y) : x^2 + y^2 \leq 1\} \), \( C = \{(x,y) : x^2 + y^2 = 1, y \geq 0\} \), \( \gamma_1(t) = \langle t, 0 \rangle, \ t \in [-1,1], \) and \( \gamma_2(t) = \langle t, \sin(\pi t) \rangle, \ t \in [-1,1]. \)

1. Sketch each object

\( \mathbf{F} \cdot \mathbf{n} = 2x^2 - 2xy + 2y^2 \) is surface in \( \mathbb{R}^3 \)

\( \text{plot3d}(x^2 - xy + y^2, x = -4..4, y = -4..4) \)

\( \mathbf{F}(x,y) = \langle 2x - y, 2y - x \rangle \) is vector field in \( \mathbb{R}^2 \)

\( \text{fieldplot}(2x - y, 2y - x, x = -3..3, y = -3..3) \)

\( D = \{(x,y) : x^2 + y^2 \leq 1\} \) set in \( \mathbb{R}^2 \)

No Maple \( \cap \) inside unit circle

\( C = \{(x,y) : x^2 + y^2 = 1, y \geq 0\} \) curve in \( \mathbb{R}^2 \)

\( \text{implicitplot}(x^2 + y^2 = 1, x = -2..2, y = -2..2) \)

\( \cap \) circle
Panel 3

Let \( f(x, y) = x^2 - xy + y^2 \), \( F(x, y) = \langle 2x - y, 2y - x \rangle \), \( D = \{(x, y) : x^2 + y^2 \leq 1\} \),
\( C = \{(x, y) : x^2 + y^2 = 1, y \geq 0\} \), \( \gamma_1(t) = \langle t, 0 \rangle \), \( t \in [-1, 1] \), and \( \gamma_2(t) = \langle t, \sin(\pi t) \rangle \), \( t \in [-1, 1] \).

1. Sketch each object

\[ \mathbf{T}_1(t) = \langle t, 0 \rangle, \quad t \in [-1, 1] \] (parametrized curve, incl. \( \mathbf{d} \))

\[ \mathbf{T}_2(t) = \langle t, \sin(t) \rangle, \quad t \in [-1, 1] \] (parametrized curve with \( \mathbf{d} \))

\[ \text{plot}([t, \sin(t), t = -1..1]); \]

Panel 4

Let \( f(x, y) = x^2 - xy + y^2 \), \( F(x, y) = \langle 2x - y, 2y - x \rangle \), \( D = \{(x, y) : x^2 + y^2 \leq 1\} \),
\( C = \{(x, y) : x^2 + y^2 = 1, y \geq 0\} \), \( \gamma_1(t) = \langle t, 0 \rangle \), \( t \in [-1, 1] \), and \( \gamma_2(t) = \langle t, \sin(\pi t) \rangle \), \( t \in [-1, 1] \).

a) \[ \int_C f(x, y)\, ds \] or \( \int_C f(x, y)\, ds \)

b) \[ \int_D f(x, y)\, ds \]

c) \[ \int_C f(x, y)\, ds = \int_{\gamma_1(t)} \left( x - y \right)\, ds = \int_{\gamma_2(t)} \left( x - y \right)\, ds \]

d) \[ \int_C f(x, y)\, dx \quad \int_C f(x, y)\, dy \]

\[ \int_C f(x, y)\, ds = \int_{\gamma_1(t)} \left( x^2 - y^2 \right)\, ds = \int_{\gamma_2(t)} \left( x^2 - y^2 \right)\, ds \]

i) \[ \int_C f(x, y)\, dr \]

j) \[ \int_C f(x, y)\, dr \]

k) \[ \int_C f(x, y)\, dr \]

\[ \mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle, \quad t \in [0, \pi] \]

\[ \int_C f(x, y)\, ds = \int_{\gamma_1(t)} \langle x, y \rangle\, ds = \int_{\gamma_2(t)} \langle x, y \rangle\, ds \]

\[ \int_{D_1} (x^2 - y^2)\, dx + \int_{D_2} (x^2 - y^2)\, dy \]

\[ \int_{D_1} \langle \cos(t), \sin(t) \rangle\, dt = \int_{D_2} \langle \cos(t), \sin(t) \rangle\, dt \]

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Panel 5

\[ f(x, y) = x^2 - xy + y^2 \Rightarrow \frac{\partial f}{\partial y} = -x \]
\[ = \frac{\partial f}{\partial x} = 2x \Rightarrow (0, 0), \frac{\partial f}{\partial x} \in \mathbb{C} \]

\[ \int_{Y_1} f(x, y) \, dx = \int_{Y_1} x^2 - xy + y^2 \, dx = \int_{-1}^{1} x^2 \, dx = \frac{1}{3} x^3 \bigg|_{-1}^{1} = \frac{2}{3} \]

\[ \int_{Y_2} f(x, y) \, dy = \int_{Y_2} (t^2 + \sin(t) + \sin^2(t)) \, dy \bigg|_{-1}^{1} = \]
\[ = \int_{-1}^{1} (t^2 + \sin(t) + \sin^2(t)) \cos(t) \, dt \]

Panel 6

Note: \[ \int_{C} \vec{F} \cdot d\vec{\ell} \] is important because it gives "work".

Because it gives "work".

Work of moving something around path \( C \) in a field \( \vec{F} \).

Need \[ \int_{C} \vec{F} \cdot d\vec{\ell} \] shortcut
Panel 7

**Fundamental Theorem for Line Integrals**

\[ \int_{\gamma} F \cdot dr = \int_{a}^{b} f(b) - f(a) \]

If \( F \) is conservative with potential function \( f \), and \( \gamma(t) \), \( a \leq t \leq b \), a smooth curve. Then:

\[ \int_{\gamma} F \cdot dr = f(b) - f(a) \]

Potential at end and potential at start.

How to tell: \( F(x,y) = (M,N) \) conservative?

\( F \) is conservative if \( \nabla \cdot F = \sum_{i=1}^{2} \left( \frac{\partial F_i}{\partial x_i} \right) \), i.e.

\[ M_x = N_y \quad \text{and} \quad N_x = M_y \]

Conservative if \( \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \)

Panel 8

Which of the following vector fields is not conservative:

(a) \( F(x,y) = (x,y) \) conservative

(b) \( F(x,y) = (x^2 + y^2, 2xy) \) conservative

(c) \( F(x,y) = (e^x \cos(y), -e^x \sin(y)) \)

\( \frac{\partial}{\partial y} (e^x \cos(y)) = e^x \cos(y) \)

\( \frac{\partial}{\partial x} (e^x \sin(y)) = e^x \sin(y) \)

Not conservative.

(d) \( F(x,y) = (x^2 \cos(y), -y^2 \sin(x)) \)

\( \frac{\partial}{\partial x} (x^2 \cos(y)) = 2x \cos(y) \)

\( \frac{\partial}{\partial y} (x^2 \cos(y)) = -x^2 \sin(y) \)

Conservative:

\( \frac{\partial}{\partial y} (x^2 \cos(y)) = \frac{\partial}{\partial x} (-y^2 \sin(x)) \)

\( \frac{\partial}{\partial x} (y^2 \sin(x)) = \frac{\partial}{\partial y} (-y^2 \sin(x)) \)

\( \frac{\partial}{\partial y} (x^2 \cos(y)) = -x^2 \sin(y) \)

\( \frac{\partial}{\partial x} (-y^2 \sin(x)) = -y^2 \cos(x) \)
Panel 9

Find \( \int \nabla \cdot \mathbf{F} \, d\mathbf{r} \) where \( \mathbf{F}(x,y) = (x^2 + y^2, 2xy) \)

\( \Gamma_1 \) and \( \Gamma_2 = \)

\[ \int_{\Gamma_1} \nabla \cdot \mathbf{F} \, d\mathbf{r} \]

\[ \int_{\Gamma_2} \nabla \cdot \mathbf{F} \, d\mathbf{r} \]

Old way:

\[ \int_{\Gamma_1} \mathbf{F} \cdot d\mathbf{r} = \int \left( x^2 + y^2, 2xy \right) \, (dx, dy) = \]

\[ \Gamma_1: \cos(t), \sin(t) \]

\[ = \int_0^\pi \left[ \cos^2(t) + \sin^2(t) \right] \, dt + \int_0^\pi 2\cos(t)\sin(t) \, dt \]

New way:

use Maple

\[ \int_0^\pi \left[ -\sin(t) \, dt + \int 2\sin(t) \cos^2(t) \, dt \right] = \frac{\pi}{2} \]

Panel 10

\[ \int_{\Gamma_2} \mathbf{F} \cdot d\mathbf{r} = \int x^2 + y^2 \, dx + 2xy \, dy = \int (-2t) \, dt = \int_0^1 \left[ \cos^2(t) + \sin^2(t) \right] \, dt + \int_0^\pi 2\cos(t)\sin(t) \, dt \]

New way:

Is \( \mathbf{F}(x,y) = (x^2 + y^2, 2xy) \) conservative?

Potential function is \( f(x,y) = \frac{1}{2}x^2 + xy \) (I guessed)

Now:

\[ \int_{\Gamma_2} \mathbf{F} \cdot d\mathbf{r} = f(\text{end}) - f(\text{start}) = \]

\[ = f(1,0) - f(1,0) = \]

\[ = -\frac{1}{2} - \frac{1}{2} = -\frac{2}{2} = -1 \]
Panel 11

**Consequences of Fundamental Theorem for Line Integrals**

**Path Independence:** If $C_1$ is a curve from $A$ to $B$, and $C_2$ is another curve from $A$ to $B$, and if $\vec{F}$ is a conservative vector field, then

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r},$$

i.e.

Line integral is independent of curve from $A$ to $B$.

Panel 12

**Ex:** Find work done by gravitational field

$$\vec{F}(\vec{r}) = -\frac{mG \vec{r}}{\|\vec{r}\|^3}$$

moving particle from $(3,4,12)$ to $(2,2,0)$.

$$\vec{F}(3,4,12) = -\frac{mG(3,4,12)}{\|(3,4,12)\|^3}$$

is conservative

with potential function

$$-mG \left(\frac{1}{(3,4,12)} - \frac{1}{(2,2,0)}\right) = f$$

$$\int_{(3,4,12)}^{(2,2,0)} \vec{F} \cdot d\vec{r} = f(2,2,0) - f(3,4,12) = mG \left[\frac{1}{8} - \frac{1}{\sqrt{97}}\right]$$

Only makes sense if $\vec{F}$ is conservative!
**Corollary 2:** If \( \mathbf{F} \) is conservative and \( C \) is a closed curve, then \( \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \).

Note: if \( C \) is closed we write \( \oint_C \mathbf{F} \cdot d\mathbf{r} \).

Because \( \mathbf{F} \) conserv., closed curve has some
same

\( f \), \textit{finish physics}

\[ \left\{ \begin{array}{l}
\int_C \mathbf{F} \cdot d\mathbf{r} = f(\text{end}) - f(\text{start}) = 0 \\
\text{same}
\end{array} \right. \]

---

**How to find a Potential Function**

\( \mathbf{F}(x,y) = \langle M, N \rangle \) a vector field.

**Wanted:** \( f(x,y) \) s.t. \( \nabla f = \mathbf{F} \) (if \( \frac{\partial N}{\partial y} = \frac{\partial M}{\partial x} \)).

1. \( f_x = M \Rightarrow f = \int M \, dx \) (anti-deriv. with \( x \))

   \( f \) will include a function of \( y \), \( C(y) \).

2. Use \( f \) in 1 and compute \( f_y \). Compare

   \( f_y = \ldots \Rightarrow C'(y) = N \).

3. Solve equation 2 for \( C(y) \) by integration

4. Check your answer
Find potential function for $f = \left( \begin{array}{c} 3x + 2xy \vspace{.1cm} \\
3 + 3y^2 \end{array} \right)$ if exists

**Panel 15**

1. \( f_x = 3 + 2xy \) \hspace{.5cm} f = \int 3 + 2xy \, dx = 3x + x^2y + C(y) \\
2. \( f_y = x^2 + C'(y) = x^2 - 3y^2 \) \\
   \hspace{.2cm} C'(y) = -3y^2 \\
3. \( C(y) = -y^3 + c \) \\
4. \( f(x,y) = 3x + x^2y - y^3 + C \)

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**Panel 16**

Find potential function for \( \langle x^2 \cos(y), -y^2 \sin(x) \rangle \)

1. \( \frac{\partial}{\partial x} f = x^2 \cos(y) \)
   \hspace{.5cm} f = \int x^2 \cos(y) \, dx = \frac{1}{3} x^3 \cos(y) + C(y) \\
2. \( \frac{\partial}{\partial y} f - f_y = -\frac{1}{3} x^3 \sin(y) + C'(y) = -y^2 \sin(x) \)

\underline{Solve:} \hspace{.2cm} Need \( C'(y) = \_\_\_ \_ \_ \_ \_ \)

\underline{No potential:} \hspace{.2cm} Always check \( \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \)
Panel 17

Find potential for \( f = (y^2, 2xy, e^{1z}, 3ye^{2z}) \) if exists.

1. \( f_x = y^2 \Rightarrow f(x,y,z) = xy^2 + C(x, y, z) \)

2. \( f_y = 2xy + C_y = 2xy + e^{2z} \)
   \[ \Rightarrow C_y = e^{2z} \]
   \[ \Rightarrow C(x, y, z) = ye^{2z} + D(z) \]

3. \( f_z = 3ye^{2z} + 0 = ye^{2z} \quad \Rightarrow D'(z) = 0 \quad \Rightarrow D \) is constant

\[ \Rightarrow f(x, y, z) = xy^2 + ye^{2z} + \text{constant} \]

Panel 18

If \( \mathbf{F}(x, y, z) = (H, N, P) \) is conservative then

\[ \nabla \cdot \mathbf{F} = (0, 0, 0) \]

\( \Box \): Which of the following vector fields is NOT conservative:

(a) \( \mathbf{F} = \left\langle xy - \sin(x), \frac{1}{2}x^2 - \frac{e^y}{e^x}, e^y \right\rangle \)

(b) \( \mathbf{G} = \left\langle xz, yz, xy \right\rangle \)

(c) \( \mathbf{H} = \left\langle 2xy - z^2, 2ye^x + x^4, y^2 - 2e^x \right\rangle \)
Panel 19

\[ G = (x^2, yz, xy) \text{ is conservative if } \text{curl} G = 0 \]

\[
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\partial & \partial & \partial \\
x^2 & yz & xy
\end{vmatrix}
\]

\[ \text{Slop } \Rightarrow \text{ not zero} \]

\[ H = (2xy - z^2, 2yz + x^2, y^2 - 2z x) \]

\[
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\partial & \partial & \partial \\
x^2 & yz & xy
\end{vmatrix}
\]

\[ \text{Slop } \Rightarrow \text{ not conserved!} \]

Panel 20

\[ \mathbf{F} = \left( xy \sin(z), \frac{e^y}{e^x}, \frac{e^y}{e^x} - x \cos z \right) \]

is conservative?
Panel 21

**Summary of Conservative Vector Field**

\[ \mathbf{F} = \nabla \varphi, \quad \varphi \text{ is potential function} \]

\[ \int_C \mathbf{F} \cdot d\mathbf{r} \text{ is independent of path from } A \text{ to } B \]

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \text{ for all closed curves } C \]

(If domain is simply connected) [technical]

\[ \text{curl } \mathbf{F} = 0 \iff \nabla \varphi \text{ conservative} \]

Panel 22

**Note:** Find curl \( \mathbf{F} \) if \( \mathbf{F}(x,y) = \langle M, N \rangle \)

\[ \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\partial_x & \partial_y & \partial_z \\
M & N & 0
\end{vmatrix} = \langle 0 - 0, 0 - 0, \frac{\partial N}{\partial y} - \frac{\partial M}{\partial x} \rangle 
\]

i.e. equin. \( \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \)
Panel 23

1. Find a conservative vector field that has the given potential: 
   \[ f(x, y, z) = \sin(x^2 + y^2 + z^2) \]

2. Find \( \nabla \cdot F \) and \( \text{curl}(F) = \nabla \times F \)
   \[ F(x, y, z) = x^2z, y^2x, y + 2z \]

3. Evaluate \( \int_C (x - y)dx + xdy \) if \( C \) is the graph of \( y^2 = x \) from \((4, -2)\) to \((4, 2)\)
   \[ y(4) = \left( \frac{4^2}{2} \right)^{1/2} \]

4. Find the work done by \( F(x, y, z) \) along the curve \( \langle t, t^2, t^3 \rangle \) from \((0, 0, 0)\) to \((2, 4, 8)\), where
   \[ F(x, y, z) = y, z, x \]

   Check which of the following vector fields is not conservative.
   \[ F(x, y) = 3x^2y + 2x^3 + 4y^3 > 
   F(x, y) = e^x - 3e^x \sin(y) > 
   F(x, y, z) = 8xyz, 1 - 6yz^2, 4x^2 - 9y^2z^2 > 
   
   \]

5. Show that the line integrals are independent of the path, and find their value:
   \[ \int_{(3,1)}^{(3,2)} (3x^2 + 2y)dx + (x^2 + 2xy)dy = f(3, 1) - f(3, 2) \]
   \[ \int_{(1,1)}^{(1,3)} (6x^3 + 2x^2)dx + (9x^2 y^2)dy + (4x + 1)dz \]

   is \( \text{curl}(F) = 0 \)

Panel 24

Find work done by \( F = \langle x^2 + y^2, 2x + y \rangle \) from \((-1, 0)\) to \((0, 1)\)

\[ \int_{-1}^{0} \frac{2 - 3}{F} \left( \begin{array}{c} 2x \cr x^2 + y^2 \end{array} \right) \, dx \]

\[ \text{Must use this} \]

\[ f_{\text{end}} - f_{\text{start}} \]

Next \(\Rightarrow \) \( f \) better be \( \text{conservative} \)

\[ F = \langle 3x^2 y + 2, x^3 + 4y^3 \rangle \] find potential:

\[ f_x = 3x^2 y + 2 \Rightarrow f = x^3 y + 2x + C(y) \]

\[ f_y = x^3 + C'(y) = x^3 + 4y^3 \Rightarrow C'(y) = 4y^3 \]

\[ \Rightarrow C(y) = y^4 + c \]

\[ c(x, y) = x^3 y + 2x + \frac{y^4}{4} + c \]
Panel 25

\[ \int \mathbf{F} \cdot d\mathbf{a} \text{ important because it is work} \]

\[ \int_{\gamma} \mathbf{F} \cdot d\mathbf{r} \text{ long way, using curve parametrization} \]

\[ \int_{\gamma} \mathbf{F} \cdot d\mathbf{r} = f(\beta) - f(\alpha) \text{ if conservative} \]

\[ \oint_{\gamma} \mathbf{F} \cdot d\mathbf{r} \text{ other shortcut!} \]

\[ \oint_{\gamma} \mathbf{F} \cdot d\mathbf{r} = \oint_{R} \nabla \times \mathbf{F} \cdot d\mathbf{A} \]

Panel 26

Green's Theorem: \( R \) a region in \( xy \)-plane with boundary curve \( C \). \( C \) is piecewise smooth, non-intersecting, closed, and positively oriented. \( \mathbf{F} = (M, N) \) is a smooth vector field. Then

\[ \oint_{C} \mathbf{F} \cdot d\mathbf{r} = \iint_{R} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA \]

Corollary: If \( \mathbf{F} \) is conservative then \( \oint_{C} \mathbf{F} \cdot d\mathbf{r} = 0 \)

already knew that anyway!

either work line integral or double integral

(it curve is closed)
Ex: Evaluate $\oint_C xy \, dx + x^3y \, dy$, where $C$ is as shown.

Try Green's Theorem: curve closed.

$$\oint_C xy \, dx + x^3y \, dy = \iint_R \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \, dA = \iint_R 3x^2 - 5x \, dy \, dx = \frac{27}{4}$$

Ex: Evaluate $\oint_C 2xy \, dx + (x^2 + y^2) \, dy$, $C$ in $4x^2 + 9y^2 = 36$.

We curve the "long way": need parametrization.

$r(t) = \langle \cos t, \sin t \rangle$

Closed curve - by Green:

$$\oint_C 2xy \, dx + (x^2 + y^2) \, dy = \iint_R \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \, dA = -\iint_{ellipse} 2x - 2x \, dA = 0$$
Coming Attractions:

Slope's + Gauss Thin

+ Review

Tomorrow