**Metric Spaces**

*Introduction and examples*

We have already studied “measure space” , i.e. a set, or better a space X together with a collection of measurable subsets and a measure *m* defined on that collection of sets.

Now will define a *metric* space, check out its properties, and learn about some examples.

**Definition: Metric Space**

A Metric Space is a pair , where is a set and is a function defined on with the following properties:

1. and implies that

**Examples**: The classical example of a metric space is, of course, our familiar , i.e. the set of real numbers with the metric function being the absolute value. But there are many other examples. Each of the following are metric spaces:

* , where
This space is called
* , where
This space is called
* , where
This space is called

So, the space can have several metrics. How can you visualize them?

**Definition: (Unit Ball)**

In a metric space the set of all that are at most 1 unit away from the origin, i.e. such that is called the unit ball for that metric.

**Example**: Draw a picture of the unit ball for our standard metric space .

We need to find all such that , or . That means that the unit ball in is the closed interval

**Example**: Show that , , and are really the same spaces.

Clearly the underlying set for the three metric spaces is the same (namely ). But the various metrics are also the same:

: so that for n = 1 we have:

: so that for n = 1 we have

: left to the reader

Thus, the three metric spaces are the same (if n = 1).

If n > 1, the metric spaces that , , and are different, which you could confirm by drawing the unit ball for each as an exercise. Lets see what the unit ball in looks like:

This looks like our “standard round” ball of radius 1, centered at the origin, which means that is our standard Euclidian, 3D space.

The tricky property for a metric is usually the triangle inequality. For the triangle inequality is based on the

**Theorem: (Cauchy-Schwartz Inequality)**

This inequality seems rather technical to prove, which is true, but there are (at least) two proofs: one rather straightforward and second, sweet and short and smart, proof.

**Proof**: First, let us verify that

We start with the term on the right:

Let’s consider the middle term:

The first term can be simplified to:

Similarly, the last term can be rewritten as:

Taking everything together we get:

or equivalently:

But since the last term is definitely positive, this implies the Cauchy-Schwartz inequality:

There is a much smarter proof which proves the Cauchy-Schwartz inequality in a couple of lines:

Consider the function:

Hence, we have a quadratic function . That means that since the quadratic function has a most one zero, the discriminant of this function . But that implies that

which is equivalent to the Cauchy-Schwarz inequality.

So far our metric spaces are variations of . But metric spaces can be more abstract:

**Example**: Consider the space of all functions continuous on the interval Add a metric to this space by defining . Then that space is a metric space, denoted by .

Similarly to putting different metrics on we can define:

**Example**: Consider the space of all functions continuous on the interval . Define a metric on this space be defining

Then this space, denoted by , is another metric space.

The proof of the triangle inequality depends on the Cauchy-Schwartz i*ntegral* inequality:

**Proposition:** **Cauchy-Schwartz Integral Inequality**:

Assuming that all the integrals exist, we have:

The proof is left as an exercise.

There are other named inequalities similar to the Cauchy-Schwartz inequality:

**Proposition: Minkowski Inequality**

* For sums/vectors: If a we have: for any p > 1
* For integrals/functions: for any

The final inequality that is important to know is:

**Proposition: Hoelder’s Inequality**

Both Minkovski and Hoelder’s inequality are used to prove the triangle inequality for additional vector spaces.

**Exercises:**

1. Draw he unit balls for , for , and for . For extra credit, can you *describe* the unit balls in the three n-dimensional spaces, perhaps in words?
2. Show that , , and really are metric spaces
3. Draw or describe the unit ball for
4. Find the distance between and : in and in
5. Prove the Cauchy-Schwartz Integral inequality *(Hint: try to use a ‘smart’ proof similar to the smart prove of the regular Cauchy-Schwartz inequality)*
6. Show that is a metric space (hint: use the Cauchy-Schwartz integral inequality)
7. Verify Minkowski’s as well as Hoelder’s inequality for and
8. Show that if  are proportional, then Minkowski’s inequality turns into an equality
9. Show that with , ,is a metric space, denoted by (Note that we already have introduced for , and .