**Lebesgue Integration**

 *Part 4: The General Lebesgue Integral*

So far we have defined the Lebesgue integral of a bounded function. That was relatively easy but if that was all that there was we wouldn’t have gained that much over Riemann integrable functions, which also have to be bounded. To define the Lebesgue integral for a general function, including one that could be equal to infinity over parts of its domain, we first need to explore measurable functions.

Recall that in Topology we learned that a function f is continuous if and only if the inverse image of every open set is open. In fact, that makes for a much better general definition of continuous function than our usual definition, which is tied to the real line, whereas the alternate definition would work between any spaces where open sets are defined. Based on that idea, we define measurable functions as functions whose inverse image of every measurable set is measurable.

**Definition (Measurable Function)**

Suppose and are two measure spaces. Then a function is called measurable if the inverse image of every measurable subset is a measurable set .

As it turns out, measurable functions are about the most general classes of functions you want to consider in analysis. Basically, anything that can be described is a measurable function, and functions that are not measurable are considered pathological in analysis. Certainly, all continuous functions on the real line are Lebesgue measurable (see exercises).

It may be difficult to check the inverse image of every measurable set, so here is an easier characterization of measurable functions:

**Proposition**: **Proof 1:** First we will show that if any one of , , , or is measurable for all ., then all of these sets are measurable.

The sets and are complements of each other, so they are equivalent because a set is measurable if and only if its complement is measurable.

Similarly you can see that and are equivalent.

Since proves that if is measurable, then is measurable, since the countable union of measurable sets is measurable, and for the other direction we note that . Thus and are equivalent.

But that means that all four conditions are equivalent.

Now let us prove if f is measurable, then is measurable for all : if is measurable, then is also measurable because (-∞,) is a Borel set, hence it is measurable, and therefore its inverse image is measurable. On the other hand, if is measurable for all , then is measurable, so that the inverse image of all open intervals are measurable because . But that implies that the inverse image of every Borel set is measurable, which implies that the function is measurable.

Measurable functions are closed under addition, multiplication, and taking limits (but not composition).

**Proposition (Combining Measurable Functions)**

If and are two measurable functions, and is a sequence of measurable functions defined on the same domain, then:

* , , and (where defined) are also measurable
* , , , , and - if it exists - are measurable

Here is the connection between measurable and simple functions

**Proposition: (Simple Functions are Measurable)**

A simple function that takes no more than countably many distinct values is measurable if and only if the sets

are measurable for all n.

**Proof 2:** First, suppose that is a measurable simple function. Since is measurable (measure zero) and is measurable, it follows that is measurable by the definition of a measurable function. Conversely, suppose the sets are measurable. Let be an arbitrary union of any of the countably-many values , so it follows that is measurable (countable union of measurable sets is measurable). The preimage of under , likewise, can be expressed as the union of any of the countably many sets , which must also be measurable (countable union of measurable sets is measurable). Therefore, is measurable by definition

**Proposition: (Measurable Functions and Simple Functions)**

A bounded function f is measurable if and only if f can be represented as the limit of a uniformly convergent sequence of simple functions.

**Proof 3:** If is the uniform limit of a convergent sequence of simple functions, then is measurable by the theorem that states that be a sequence of measureable functions on *X,* and let be a function on *X* such that for every Then is itself measurable, since simple functions are measurable by definition.

On the other hand, take any bounded measurable function , and assume for all . Let

Then, because is measurable, the sets are measurable, so that the functions are simple and converge uniformly to as since

As it turns out, sequences of functions that converge pointwise on a set can be made to converge uniformly on a subset of with almost the same measure as (compare to Littlewood’s Three Principles):

**Proposition (Egorov’s Theorem)**

Let be a sequence of measurable functons converging almost everywhere on a measurable set to a function . Then, given any there exists a measurable set such that:

1. converges uniformly on to f

**Proof** The function f is measurable since it is the limit of measurable functions*.* Let

 (1)

Therefore if *m* and *n* are fixed, is the set of every point x such that

holds for all . Furthermore, let’s have

Looking back at (1), it follows that

Since this is an increasing sequence of measurable setswe have

Therefore, given any and any δ > 0, there is an such that

. (2)

Now let

Then satisfies both of the conditions of Egorov’s theorem: If , then, given any m =1,2,3,…

for every This shows that the sequence { is uniformly convergent on

In order to show , let’s take the measure of the set -.. If , then there are arbitrarily large values of such that

meaning that will not converge to *f* at the point *x0*. Thus , since converges to *f* almost everywhere, by assumption. From (2) it follows

Therefore

and therefore .

Sources: Introductory Real Analysis by A.N. Komogorov & S.V. Fomin, p???

Also – again, compare to Littlewood’s Three Principles – measurable functions are almost continuous, or more precisely: a measurable function can be made continuous by changing it on a set of arbitrarily small measure:

**Theorem (Luzin’s Theorem)**

If is a function defined on a closed interval , then is measurable if and only if given any there is a continuous function such that .

**Proof :** First, we will show that ifis simple and then there exists a closed set with m(E \ F) < such that is continuous, where is the restriction of to

Because is simple where are coefficients and are disjoint measurable sets. Since each is measurable, we can find closed sets with m( for . Now let F is closed as a finite union of closed sets, m(E\F), and restricted to - or - is continuous.

Now we show that for any measurable function and there is a closed set such that m( and is continuous where is the restriction of to , which is, essentially, Lusin’s theorem

Let be a sequence of simple functions that converge pointwise to on . By the first part of this proof we have that for each we can pick a closed set with m( \ such that the restriction of to is continuous.

By Egorov’s Theorem we know that there is a such that m(B and converges uniformly to on \. Without loss of generality, we can assume that is open (why). Finally, let which is closed since B was assumed to be open. .

We know that the restriction of to is continuous because a uniformly convergent sequence of continuous functions is continuous.

But which proves the theorem

Note: with a little more effort we can show that Lusin’s theorem is true for measurable.

Source: <https://www.ma.utexas.edu/users/lpbowen/m381c/lecture-notes.pdf>

Now we (finally) have all the ingredients to define the general Lebesgue integral

**Definition (General Lebesgue Integral)**

A measurable function defined on a measurable set is called Lebesgue integrable if there exists a sequence of integrable simple functions converging uniformly to on . The limit

is called the Lebesgue integral of over the set , written as

This definition relies on the following conditions:

1. The limit exists and is finite for any uniformly convergent sequence of integrable simple functions
2. The limit is independent of the choice of the sequence
3. The limit reduces to the previously defined Lebesgue integral for simple and for bounded functions

Condition 1 follows from the estimate

As for condition 2, suppose and both converge uniformly to but

Consider the sequence . Then converges uniformly to but does not exist, contradicting condition (1).

Condition 3 is left as exercises.

**Proposition (Properties of Lebesgue Integral)**

If and are integrable over a set then

1. For any number we have is integrable and
2. The function is integrable and
3. If almost everywhere then
4. If and are disjoint measurable sets contained in E, then

**Proof:**

*(I do not understand this proof):*

We have since almost everywhere in. Thus . By (ii) have. Hence

*(I do not understand this proof)*:

One reason why the Lebesgue integral is preferred over the Riemann integral is the easy of switching integration and limits:

**Theorem (Lebesgue Dominated Convergence Theorem)**

Let be integrable over and a sequence of functions converging to a limit on such thaton . Then is integrable and

**Proof: 7** TBD

Here is another theorem that can be used to switch integration and limits:

**Theorem (Lebesgue Monotone Convergence Theorem)**

If is a sequence of integrable functions such that , then the limit exists and is integrable with

**Proof 8:** Because is monotonically increasing:

By Fatou’s Lemma:

Since the inf is an upper bound and the sup is a lower bound:

Note: This theorem is also known as **Levi’s** theorem.

Finally, there is one more important theorem we want to state for the record.

**Theorem (Fatou’s Lemma)**

If is a sequence of non-negative measurable functions and almost everywhere on a set then

**Proof 9:** (*This proof needs to be redone or explained better)*

The proof uses the monotone convergence theorem:

For every natural number k, define point wise the functions

Then the , , … form an increasing sequence of measurable functions, meaning that , for all k, and converges point wise to the lim inf f. For all k we have , so that by the monotonicity of the integral hence .

By the monotone convergence theorem

**Exercises:**

1. Prove that every continuous function is Lebesgue measurable.
2. Prove that every bounded, measurable function f on a measurable set E with finite measure is Lebesgue integrable
3. Show that the general definition of Lebesgue integration reduces to prior definitions if is bounded or simple.
4. Show that if is a sequence of integrable functions such that for all on a measurable set and, then
5. Is the above statement true for Riemann integrable functions and Riemann integrals?