Homework 5 (Lebesgue Measure)

This time I want you to work out all of the problems listed below and submit your solutions as a homework set. It is due the Monday after our Spring Break (3/13)

1. What is the cardinality of all Lebesgue measurable subsets of $R$**.***Hint: Find an uncountable set with measure zero. Then recall that every set with measure zero is measurable.*
2. If $X=\{a, b, c, d\}$, is $Σ = \{ ∅, \{a, b\}, \{c, d\}, \{a, b, c, d\} \}$ a σ-algebra on $X$? Define a probability measure on $Σ$. *Hint: This is straight-forward. Note that there are many ways to define a probability measure; you just need to specify one measure.*
3. Let $\{E\_{n}\}$ be a countable collection of sets and recursively define sets $F\_{n}$ as follows: $F\_{1}=E\_{1}$ and $F\_{n}=E\_{n}-\left(E\_{1}∪E\_{2}∪…∪E\_{n-1}\right)$ for $n > 1$. Show that the $F\_{n}$ are disjoint and the union of the $F\_{n}$ is the same as the union of the $E\_{n}$. *Hint:* *Straight-forward*
4. Show that (countable) additivity implies monotonicity. *Hint: if* $A⊂B$ *then* $B=A∪(B-A)$
5. Show that countable additivity implies countable subadditivity. *Hint: Use one of the previous problems.*
6. Show that in the definition of a measure we could also assume that there exists at least one set E with $m\left(E\right)<\infty $ instead of that $m\left(∅\right)=0$. *Hint: In other words, you need to show that if* $m\left(E\right)<\infty $ *than* $m\left(∅\right)=0$
7. Define a function $f$ on the set $X=\{a, b, c\}$ by setting$ f\left(O\right)=0$, $f\left(X\right)=2$, and $f(A)=1$ for any other subset $A$ of $X$. Could this function be called an outer measure? Is it additive? *Hint. Check the various conditions for a measure.*
8. Show that the Cantor middle-third set $C\_{3}$ is measurable and find its measure. Note that we already did this by finding the ‘length’ of the Cantor set, but strictly speaking the concept of length only applies to intervals, not to the Cantor set. But the concept of measure does apply, so … what is $m(C\_{3})$
9. What is $m(C\_{5})$, i.e. the measure of Cantor’s middle fifth set?
10. Show that very compact subset of the real numbers is L-measurable with finite measure. *Hint: Compact means what for subsets of the real line?*
11. For any two sets A, B ⊂ **R**, prove: $m\left(A\right)+m(B)\leq 2m(A△B)+ 2m(A∩B)$, where
$A△B=\left(A∪B\right)-\left(A∩B\right)=\left(A-B\right)∪(B-A)$ is the symmetric difference between A and B. *Hint: Try using the fact that* $A=\left(A-B\right)∪(A∩B)$ *and* $A-B⊂A△B$*.*
12. Suppose A ⊆ E ⊆ B, where A and B are measurable sets of finite measure. Prove that if m(A) = m(B), then E is measurable. *HJnt: A set with measure zero is measurable, and* $E-A⊂B-A$
13. Prove that if A is L-measurable then for any $ϵ>0$ there exists an open set $O$ such that $A⊂O$ and $m\left(O-A\right)<ϵ$ *Hint: Follows directly from the definition of L-outer measure and the properties of the inf*
14. Prove that if A is L-measurable then for any $ϵ>0$ there exists a closed set $F$ such that $F⊂A$ and $m\left(A-F\right)<ϵ$. *Hint: A is measurable implies that* $A^{c}$ *is measurable. Then apply the above property to* $A^{c}$*.*
15. Prove that given $ϵ>0$ and A L-measurable, there exists an open set O and closed set F so that $F⊂A⊂O$ and $m\left(A-F\right)<ϵ$ and $m\left(O-F\right)<ϵ$
16. Prove that the Lebesgue outer measure is not additive. *Hint: If for every disjoint sets A and B the Lebesgue measure was additive, then every set would be L-measurable.*
17. In the proposition on monotone sequences of decreasing sets we had to assume that $m\left(A\_{1}\right)$ was finite. Show that without this assumption the statement in that proposition is false.

BONUS: Suppose m is Lebesgue measure. Define x + A = {x + y : y ∈ A} and cA = {cy : y ∈ A} for x, c ∈ R. Show that, if A is a Lebesgue measurable set, then m(x + A) = m(A) and m(cA) = |c|m(A).

PROJECT: Prove that there exists a subset of the reals that is not Lebesgue measurable.