**Power/Taylor Series**

*Part 4: We have a problem*

We have a few basic power series at our disposal and we have learned some tricks to find more series. However, each of our examples *presumed* that a given function *had* a power series and then proceeded from there. But this is not necessarily true: while it is clear that if a function $f(x)$ can be written as a power series, it must be $C^{\infty }$ (i.e. infinitely often differentiable) the converse is not necessarily true:

**Theorem**: If a function $f\left(x\right)$ can be written as a power series with a radius of conversion $r>0$ then $f$ is necessarily $C^{\infty }$

Example: Let $f\left(x\right)=\left\{\begin{array}{c}e^{-\frac{1}{x^{2}}}, \&x\ne 0\\0 , \&x=0\end{array}\right.$. Find the power series expansion for this function $f(x)$ centered at zero, assuming there was one.

We need to find $f^{\left(n\right)}\left(0\right)$ for all n. Clearly $f\left(0\right)=0$, so let’s find the first derivative at zero:

$$f^{'}\left(0\right)=\lim\_{h \to 0}\frac{f\left(0+h\right)-f(0)}{h}=\lim\_{h \to 0}\frac{e^{-\frac{1}{h^{2}} }}{h}=\lim\_{h\to 0}\frac{1/h}{e^{1/h^{2}}}$$

Now we substitute $\frac{1}{h}=u$ which goes to infinity as h goes to zero, so that

$$f^{'}\left(0\right)=\lim\_{h\to 0}\frac{1/h}{e^{1/h^{2}}}=\lim\_{u\to \infty }\frac{u}{e^{u^{2}}}=\lim\_{u\to \infty }\frac{1}{2u e^{u^{2}}}=0$$

where we have used l’Hospital rule to find the limit. Thus, we have:

$$f'\left(x\right)=\left\{\begin{array}{c}\frac{2}{x^{3}}e^{-\frac{1}{x^{2}}}, \&x\ne 0\\0 , \&x=0\end{array}\right.$$

But then

$$f^{''}\left(0\right)=\lim\_{h\to 0}\frac{f^{'}\left(0+h\right)-f'(0)}{h}=\lim\_{h \to 0}\frac{\frac{2}{h^{3}}e^{-\frac{1}{h^{2}} }}{h}=\lim\_{h\to 0}\frac{\frac{2}{h^{3}}\frac{1}{h} }{e^{1/h^{2}}}=\lim\_{u\to \infty }\frac{2u^{4}}{e^{u}^{2}}=0$$

where we have again used l’Hospital rule multiple times to find the limit. Similarly, we can show that $f^{\left(n\right)}\left(0\right)=0$ for all $n$ (try this by induction, see <http://www.mathcs.org/analysis/reals/cont/fp_cinf.html>). Thus, we found a function that is infinitely often differentiable but its power series expansion $p\left(x\right)=\sum\_{n=0}^{\infty }\frac{f^{\left(n\right)}\left(0\right)}{n!} x^{n}$ does not converge to the original function, since all coefficients are zero but the function itself is not identically zero.

So, we kind of have three types of power series:

1. A series $\sum\_{}^{}a\_{n}\left(x-c\right)^{n}$ with powers of $\left(x-c\right)^{n}$ and more or less arbitrary coefficients $a\_{n}$.
2. A series that starts out with a $C^{\infty }$ function $f(x)$ and whose coefficients depend on that function, as in $\sum\_{}^{}\frac{f^{\left(n\right)}\left(c\right)}{n!}\left(x-c\right)^{n}$
3. A series as in (2) but that converges back to the original function $f\left(x\right)=\sum\_{}^{}\frac{f^{\left(n\right)}\left(c\right)}{n!}\left(x-c\right)^{n}$

**Definition (Taylor Series)**

A Taylor series $T\_{f}(x,c)$ for a given function $f$ is a power series centered at $x=c$ whose coefficients are

$$a\_{n}=\frac{f^{\left(n\right)}\left(c\right)}{n!}$$

If the center $c=0$ then the Taylor series $T\_{f}(x,0)$ is traditionally called **McLaurin** series.

So, every Taylor series is a power series but not every power series is a Taylor series, and (by the above example) not every Taylor series converges to its generating function.

There are two theorems that clarify when a Taylor series converges back to $f$ – we will cover that in the last and final segment on power series. For now, we will conclude with a simple example of Taylor series:

**Theorem**: If $p(x)$ is a polynomial then any Taylor series $T\_{p}(x,c)$ converges to $p$ for any center $c$.

To illustrate, consider $p\left(x\right)=x^{3}+2x^{2}+3x+4$. Find $T\_{p}(x,0)$ and $T\_{p}(x,1)$ and show that they are the same.

First, let’s find $T\_{p}(x,0)$, i.e. the Taylor series for p centered at zero. We need to compute:

* $p\left(x\right)=x^{3}+2x^{2}+3x+4$ so that $p\left(0\right)=4$
* $p'\left(x\right)=3x^{2}+4x+3$ so that $p'\left(0\right)=3$
* $p''\left(x\right)=6x+4$ so that $p^{''\left(0\right)}=4=2∙2!$
* $p'''\left(x\right)=6$ so that $p^{'''\left(0\right)}=6=1∙3!$
* $p^{\left(4\right)}\left(x\right)=0$ so that $p^{\left(4\right)}\left(0\right)=0$ and all higher derivatives are zero as well.

So:

$$T\_{p}\left(x,0\right)=4+3x+\frac{4}{2!}x^{2}+\frac{6}{3!}x^{3}=4+3x+2x^{2}+x^{3}$$

To find $T\_{p}\left(x,1\right)$ we compute the derivatives of $p$ at $x=1$:

* $p\left(1\right)=1+2+3+4=10$
* $p^{'}\left(1\right)=3+4+3=10$
* $p^{''}\left(1\right)=10$
* $p^{'''}\left(1\right)=6$

So:

$$T\_{p}\left(x,1\right)=10+10\left(x-1\right)+\frac{10}{2!} \left(x-1\right)^{2}+\frac{6}{3!} \left(x-1\right)^{3}=10+10 \left(x-1\right)+5\left(x-1\right)^{2}+\left(x-1\right)^{3}$$

Now it is easy to see that $T\_{p}\left(x,0\right)=T\_{p}\left(x,1\right)=p(x)$

Note that we will introduce a general theorem that shows when a Taylor series converges to its original function in the next segment. Then we can easily prove the above theorem (as an exercise) but for now we’ll skip the proof and are happy with the above illustration.

Exercises:

1. True or false: If $f$ and $g$ are two $C^{\infty }$ functions such that $f^{\left(n\right)}\left(c\right)=g^{\left(n\right)}(c)$ for all $n$ then $f\left(x\right)=g(x)$ for all x.
2. Is every McLaurin series a Taylor series? How about the other way round?
3. Does every Taylor series represent its original function? Prove it or give counter example.
4. Use the above theorem to prove the factor theorem for polynomials, which states that $(x-c)$ is a factor of the polynomial $p(x)$ if and only if $p\left(c\right)=0$