**Power / Taylor Series**

*Part 2: By Substitution and Basic Algebra*

We now have a few basic power series at our disposal:

* Geometric series: $\frac{1}{1-x}=\sum\_{n=0}^{\infty }x^{n}=1+x+x^{2}+x^{3}+x^{4}+…$ for $\left|x\right|<1$
* Exponential series: $e^{x}=\sum\_{n=0}^{\infty }\frac{x^{n}}{n!}=1+x+\frac{1}{2!}x^{2}+\frac{1}{3!}x^{3}+\frac{1}{4!}x^{4}+…$ for all x
* Series for sine: $\sin(\left(x\right))=\sum\_{n=0}^{\infty }\left(-1\right)^{n}\frac{x^{2n+1}}{\left(2n+1\right)!}=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}+…$ for all x
* Series for cosine: $cos⁡(x)=\sum\_{n=0}^{\infty }\left(-1\right)^{n}\frac{x^{2n}}{\left(2n\right)!}=1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}-\frac{x^{6}}{6!}+…$ for all x

From these we can figure out additional series by simple substitution.

Find a series expansion for $f\left(x\right)=\frac{1}{1+x^{2}}$.

$$\frac{1}{1+x^{2}}=\frac{1}{1-\left(-x^{2}\right)}=\sum\_{n=0}^{\infty }\left(-x^{2}\right)^{n}=\sum\_{n=0}^{\infty }\left(-1\right)^{n}x^{2n}=1-x^{2}+x^{4}-x^{6}+x^{8}-…$$

Find a series expansion for $f\left(x\right)=x^{3}e^{x^{2}}$

We know that $e^{x}= \sum\_{n=0}^{\infty }\frac{x^{n}}{n!}$. Then

$$x^{3}e^{x^{2}}=x^{3}\sum\_{n=0}^{\infty }\frac{(x^{2})^{n}}{n!}=\sum\_{n=0}^{\infty }\frac{x^{2n+3}}{n!}=x^{3}+x^{5}+\frac{x^{7}}{2!}+\frac{x^{9}}{3!}+…$$

We can of course combine substitution with integration and/or differentiation:

Find the series expansion for $f\left(x\right)=2x sin⁡(x^{2})$

We can go about this in two different ways. Method 1 starts by substituting $x^{2}$ in the series expansion for $sin(x)$, then multiplying everything by $2x$:

$$2x sin\left(x^{2}\right)=2x \sum\_{n=0}^{\infty }\left(-1\right)^{n}\frac{\left(x^{2}\right)^{2n+1}}{\left(2n+1\right)!}=\sum\_{n=0}^{\infty }\left(-1\right)^{n}\frac{2 x^{4n+3}}{\left(2n+1\right)!}=2x^{3}-\frac{2 x^{7}}{3!}+\frac{2 x^{11}}{5!}-\frac{2 x^{15}}{7!}$$

Alternatively, we could write $2x sin\left(x^{3}\right)=\frac{d}{dx}(-cos(x^{2}))$ and start with the series expansion for $cos⁡(x)$:

$$2x sin\left(x^{2}\right)=\frac{d}{dx}\left(-cos\left(x^{2}\right)\right)=\frac{d}{dx}\left(-\sum\_{n=0}^{\infty }\left(-1\right)^{n}\frac{\left(x^{2}\right)^{2n}}{\left(2n\right)!}\right)=-\sum\_{n=1}^{\infty }\left(-1\right)^{n} \frac{4n x^{4n-1}}{\left(2n\right)!}=$$

$$=\frac{4}{2!}x^{3}-\frac{8 x^{7}}{4!}+\frac{12 x^{11}}{6!}-…=2x^{3}-\frac{2 x^{7}}{3!}+\frac{2x^{11}}{5!}-…$$

Either way, the answer is the same.

**Exercises:**

1. Find the series expansion for $f\left(x\right)=\frac{2x}{\left(1-x^{2}\right)^{2}}$ and $f\left(x\right)=arctan⁡(x)$
2. Use a series computed above to prove that $π=4(1 – 1/3 + 1/5 – 1/7 + 1/9 -…)$ Note that it is interesting that the complicated transcendental number \pi is a sum of 1 over plus/minus the odd integers. However, the rate of convergence is rather slow; try to approximate $π$ to, say, 4 digits