Analysis 2 - HW6

Note Title 3/19/2013

(1) Prove the Cauchy-Schwarz inequality in $\mathbb{R}^3$; i.e.

$$\left(\sum_{k=1}^{3} a_k b_k\right)^2 \le \left(\sum_{k=1}^{3} a_k^2\right)\left(\sum_{k=1}^{3} b_k^2\right)$$

Hint: Use the same idea we used in class.

(2) Verify that

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - b_i a_j)^2$$

Deduce the Cauchy-Schwarz inequality from this.

(3) We introduced 3 versions of $\mathbb{R}^2$ in class:

$(\mathbb{R}^2, \rho) : \rho(x,y) = \sqrt{\sum_{i=1}^{2}(x_i - y_i)^2} \to \mathbb{R}^2_2$

$(\mathbb{R}^2, \rho) : \rho(x,y) = \sum_{i=1}^{2}|x_i - y_i| \to \mathbb{R}^2_1$

$(\mathbb{R}^2, \rho) : \rho(x,y) = \max_{1\le i\le 2}(|x_i - y_i|) \to \mathbb{R}^2_\infty$

In each case, describe the "unit ball" centered at 0 defined by $\rho(x,0) \le 1$

(4) Define $C_{[a,b]}$ as the set of all continuous functions $f:[a,b]\to\mathbb{R}$ with the metric

$$\rho(f,g) = \max_{a\le t\le b}|f(t) - g(t)|$$

(a) Let $f(x) = x^2 \in C_{[0,1]}$. Find $\rho(f,0)$

(b) Consider $C_{[0,2\pi]}$ and find $\rho(\sin(t), \cos(t))$.

(c) Describe the "unit ball" $\rho(f,0) \le 1$

where $f \in C_{[a,b]}$

(a) Show that $(C_{[a,b]}, \rho)$ is a metric space

(5) Show that for any metric space (X, ρ):

$$|\rho(x,z) - \rho(y,u)| \le \rho(x,y) + \rho(z,u)$$

$\forall\, x, y, z, u \in X$

(6) Show that for any metric space (X, ρ):

$$|\rho(x,z) - \rho(y,z)| \le \rho(x,y) \qquad \forall\, x, y, z \in X$$