Panel 1

Review Analysis 2

Taylor Series
Lebesgue Measure + Outer Measure
Leb. int. for bdd function

Fourier Series

Metric Spaces
complete + seperable
Baire cat. thm.

Applications:
- storing data
- RSA encryption

1

Panel 2

Taylor Series:

Power Series: $\sum_{n=0}^{\infty} a_n (x-x_0)^n \quad |x-x_0|<R$ ← radius of conv

Taylor Series for f: $\sum_{n=0}^{\infty} a_n (x-x_0)^n$

$a_n = \frac{f^{(n)}(x_0)}{n!}$

Taylor Series converges to f:
$R_{n+1}(x) = \frac{1}{n!}\int_{x_0}^{x} (x-t)^n f^{(n+1)}(t)\,dt$ if $R_n(x) \to 0 \quad n \to \infty$
Lagrange Remainder

$E=mc^2$, f n-diffble $\Rightarrow f \approx p_n(x)$, p_n polyn. of degree n

2

Panel 3

Big bad function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$f \in C^\infty$, $f^{(n)}(0) = 0 \rightarrow$ Taylor series $= 0 \neq f(x)$

Find Taylor series for: $\dfrac{x}{(1-x^2)^2}$

or $\dfrac{1}{1-x^3} = \sum_{n=0}^{\infty} x^{3n}$ or e^{2x^2}

3

Panel 4

Lebesgue Measure

(Leb.) Outer Measure

$$\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \ell(A_j) : \bigcup_{j=1}^{\infty} A_j \supset A,\ A_j \text{ open intervals} \right\}$$

(i) $\mu^*: P(\mathbb{R}) \rightarrow \mathbb{R}^+$

(ii) $\mu^*($ interval $) =$ length

(iii) monotone, i.e. if $A \subset B$ $\rightarrow \mu^*(A) \leq \mu^*(B)$

(iv) countably subadditive

(Leb.) Measurable sets: E is (C.) measurable if
$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$ $\forall A \subset \mathbb{R}$

(Leb.) Measure: $\mu(E) = \mu^*(E)$ if E (C.) msble.

4

Panel 5

Properties of Lebesgue Measure

(1) All intervals are msble and their measure is length

(2) Open + closed sets are msble

(3) Unions, intersect., complements of msble sets are msble

(4) $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{j=1}^{\infty} \mu(A_j)$

(5) $\mu\left(\bigcup A_n\right) = \sum \mu(A_j)$, A_j's are disjoint and msble

$\{f(x) \geq \alpha\}$ is msble if f cont.

5

Panel 6

$\mu^*(\emptyset) = 0$, $\mu^*(\mathbb{R}) = \infty$

$\mu^*([a,b]) = \mu^*((a,b)) = b-a$

A, B msble $\Rightarrow$ $A \cup B$ is msble

etc.

6

Panel 7

Lebesgue Integral

Characteristic function: $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{else} \end{cases}$

Simple function: $s(x) = \sum_{j=1}^{n} a_j \chi_{A_j}(x)$, A_j mble

Integral of simple function:

$$\int s(x)\,d\mu = \sum_{j=1}^{n} a_j \mu(A_j)$$

Integral of bounded function:

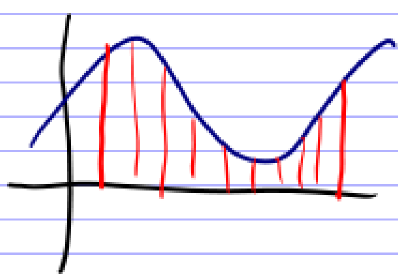
$I^*(f) = \inf\{\int s(x)\,d\mu : s \geq f\}$ $\quad I^* = I_*$ then

$I_*(f) = \sup\{\int s(x)\,dx : s \leq f\}$ $\quad f$ Leb. int'ble.

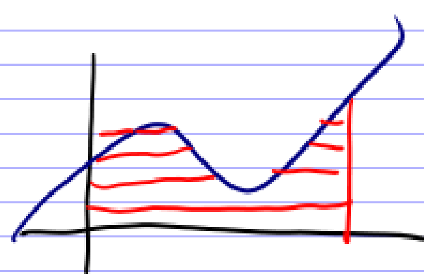
7

Panel 8

Riemann Int.



Leb. int



8

Panel 9

If $\mu(E)=0 \Rightarrow \int_E f \, d\mu = 0$, f bdd.

If f is cont., then f is (L) intble on $[a,b]$

If f is bounded and R. intble, then f is also L-intble and $\int_{[a,b]} f \, d\mu = \int_a^b f(x)\,dx$

There is a function f not L. intble (but it is tricky)

9

Panel 10

Fourier Series

The Fourier Series: $a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$

Fourier Series of f: as above

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$ $\quad a_n = \frac{1}{\pi}\int_{-\pi} \cos(nx) f(x)\,dx$

$b_n = \quad \sin \quad dx.$

Fourier Series converges - f.

f piecewise cont on $[-\pi,\pi]$, f' exists piecewise + cont then $f_n \Rightarrow f$, i.e. Fourier Series conv. uniformly!

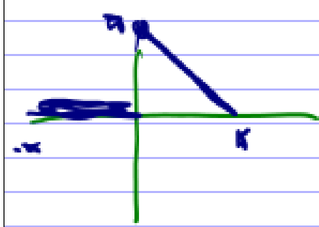
10

Panel 11

$f(x)=x$ on $[-\pi,\pi]$ ✓

$$f(x)=\begin{cases}\pi & \text{on } [0,\pi) \\ 0 & \text{on } [-\pi,0)\end{cases}$$

$$f(x)=\begin{cases}x & \text{on } [0,\pi] \\ 0 & \text{on } [-\pi,0)\end{cases}$$



? FS for this one ?

HPJ Glen.

11

Panel 12

Metric Spaces

Metric Space (X,ρ).

X set of points, $\rho: X\times X\to\mathbb{R}^+$,

$\rho(x,y)\ge 0$

$\rho(x,y)=0 \Rightarrow x=y$

$\rho(x,y)=\rho(y,x)$

$\rho(x,z)\le\rho(x,y)+\rho(y,z)$

Examples: $C^0[a,b]$, C^1, C^2

$\mathbb{R}^n_\infty$, $\mathbb{R}^n_1$, $\mathbb{R}^n_2$

ℓ_2, ℓ_p

$\ell_2=\{\{x_i\}$ s.t. $\sum|x_i|^p<\infty\}$

12

Panel 13

Inequalities

Cauchy-Schwarz: $\left(\sum a_n b_n\right)^2 \leq \left(\sum a_n^2\right)\left(\sum b_n^2\right)$

Schwarz $\left(\int fg\,dx\right)^2 \leq \left(\int f^2 dx\right)\left(\int g^2 dx\right)$

Hölder $\sum |a_n b_n| \leq \left(\sum |a_n|^p\right)^{1/p} \left(\sum |b_n|^q\right)^{1/q}$

$1/p + 1/q = 1$

Minkowski

Look it up!

13

Panel 14

Special Metric Spaces

(X, ρ) and (Y, ρ') metric. Then $f: X \to Y$ cont.:
iff $\rho'(f(x), f(x_0)) < \varepsilon$ whenever $\rho(x, x_0) < \delta$

$\{x_n\}$ sequence in (X, ρ). Then $x_n \to x_0$:
$\rho(x_n, x_0) \to 0$ as $n \to \infty$

A metric space is separable: contains countable dense subset.

A metric space is complete: if every Cauchy sequence converges in X.

14

Panel 15

$f: C^0[a,b] \to C^0[a,b]$, $f(g) = \int_a^x g(s)ds$

f onto, f 1-1, f cont?

l_2 is separable $\{r_1, r_2, r_3, \ldots, r_n, 0, 0, \ldots\}$, $r_i \in \mathbb{Q}$

$m = \{$bdd sequences$\}$, sup-norm. m is NOT separable

$C^0[a,b]$ complete

$C^2[a,b]$ not complete!

15

Panel 16

Baire Category Thm

Nested Sphere Thm: If every intersection of closed nested spheres with $r_i \to 0$ has non-empty intersection then X is complete!

A is of 1^{st} Category: A is countable union of nowhere dense sets

A is of 2^{nd} Category: not 1^{st}.

Baire Category Thm: Complete metric spaces are of 2^{nd} category

16

Panel 17

complete metric space w/o isolated points is
uncountable.

17