Panel 1

Complete Metric Spaces

Def: (X,ρ) a metric space, $\{x_n\}$ a sequence in X s.t. given $\varepsilon > 0$ $\exists N$ s.t. $\rho(x_n, x_m) < \varepsilon$ $\forall n,m \geq N$. Then $\{x_n\}$ is a fundamental, or Cauchy, sequence.

Def. (X,ρ) is complete if every Cauchy se converges to $x \in X$

Ex: $\mathbb{R}$ is complete, $\mathbb{Q}$ is incomplete, $\mathbb{N}$ complete!

1

Panel 2

$\mathbb{Q}$: $\{\frac{1}{n}\} = 1, \frac{1}{2}, \frac{1}{3}, \ldots \to 0 \in \mathbb{Q}$ Thus $\mathbb{Q}$ incomplete

$\{1, 1.4, 1.41, 1.414, 1.4142, \ldots\}$ is Cauchy, conv. to $\sqrt{2} \notin \mathbb{Q}$

HW Find a sequence defined recursively s.t.

(a) $x_n \in \mathbb{Q}$

!!!! (b) $x_n \to \sqrt{2}$ (with proof)

$\mathbb{N}$: $\{2, 4, 6, 8, \ldots\}$ not Cauchy

$\{2, 4, 6, 6, 6, \ldots\}$ is Cauchy + conv. to 6 in $\mathbb{N}$

If $\{x_n\}$ in $\mathbb{N}$ is Cauchy $\Rightarrow$ $\exists N$ s.t. $|x_n - x_m| < \frac{1}{2}$ $\forall n,m \geq N$

$\Rightarrow x_n = x_m$ $\forall n,m \geq N$. Thus, seq. conv. to $\mathbb{N}$

2

Panel 3

Is $C[a,b]$ complete?

$f: [a,b] \to \mathbb{R}$ cont, $\rho(f,g) = \max_{t \in [a,b]} \left(|f(t) - g(t)| \right)$

$\{f_n\}$ Cauchy $\Rightarrow$ (1) $|f_n(t) - f_m(t)| < \varepsilon \quad \forall\, n,m \geq N$

Thus, for t fixed, $\{f_n(t)\}$ Cauchy in $\mathbb{R}$, complete

$\Rightarrow$ $f_n(t)$ conv. pointwise to f

(1) is true uniformly $\forall t$ $\Rightarrow$ f_n conv. uniformly to f

Thus, f must be cont.

$\Rightarrow$ $C[a,b]$ is complete!

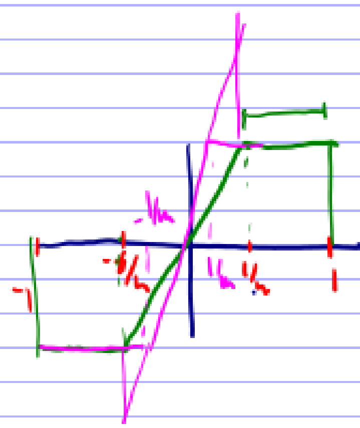
3

Panel 4

$C^2[-1,1]$ is not complete! $f: [a,b] \to \mathbb{R}$ cont.

$$\rho(f,g) = \left(\int_a^b (f(t) - g(t))^2 \right)^{1/2}$$

Let $f_n(x) = \begin{cases} -1 & , -1 \leq x \leq -\frac{1}{n} \\ nt & , -\frac{1}{n} \leq x < \frac{1}{n} \\ 1 & , \frac{1}{n} \leq x \leq 1 \end{cases}$



(1) f_n Cauchy? $\rho(f_n, f_m) \xrightarrow{?} 0$, $n \leq m$ WOLOG

$$\int_{-1}^{1} (f_n(t) - f_m(t))^2 \, dt \leq \int_{-1/n}^{1/n} (nt - mt)^2 \, dt = (n-m)^2 \frac{2}{3} \frac{1}{n^3} \to 0$$

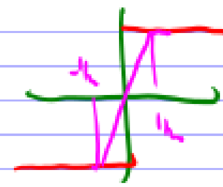
4

Panel 5

Thus, $\{f_n\}$ is Cauchy in $C^2[-1,1]$

Define

$$g(x) = \begin{cases} 1 & , 0 \le x \le 1 \\ -1 & , -1 \le x < 0 \end{cases}$$



Take any $f \in C^2[a,b]$

$$\left(\int (f-g)^2\right)^{1/2} \le \left(\int (f-f_n)^2 dt\right)^{1/2} + \left(\int (f_n-g)^2 dt\right)^{1/2}$$

$f - g \neq 0$, because f is cont. so that left side is positive

But $\lim\limits_{n\to\infty} \int (f_n - g)^2 dt = 0$

HW

Thus $\lim\limits_{n\to\infty} \int (f-f_n)^2 dt \neq 0$

$\Rightarrow C^2[-1,1]$ i/c

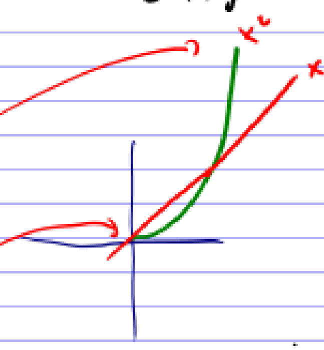
Panel 6

Q: If $f_n \to f$ in $C^2[a,b]$, does $f_n \to f$ in $C^1[a,b]$?

How about other way around?

$C^2[a,b]$: $\rho(f,g) = \left(\int_a^b (f-g)^2 dx\right)^{1/2}$

$C^1[a,b]$: $\rho(f,g) = \int_a^b |f-g| dx$



$C^0[a,b]$: $\rho(f,g) = \max(|f(t)-g(t)|)$

If $f_n - f$ is small then $|f_n - f| \ge (f_n - f)^2$

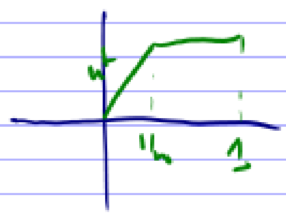
$$\Rightarrow \int_a^b |f_n - f| dt \ge \int_a^b (f_n - f)^2 dt$$

Panel 7

Thus: $f_n \to f$ in $C^1[a,b]$

$\Rightarrow f_n \to f$ in $C^2[a,b]$

Converse is false: HW



$f_n \to f(x) = 1$ in C^2

but not in C^1

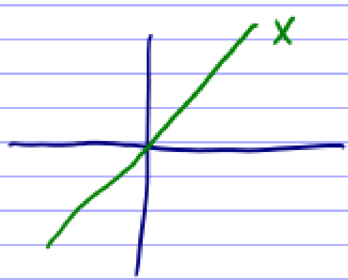
Maybe!

7

Panel 8

Consider $(\mathbb{R}, \rho)$ with $\rho(x,y) = |\arctan(x) - \arctan(y)|$

Then $(\mathbb{R}, \rho)$ is not complete!



e.g. $\{n\}$ does not conv.

show it is Cauchy! HW

Hint: MVT from Calc

8

Panel 9

Q: We discussed the shift operator on ℓ_2, which was cont.. Is it 1-1? How about onto?

ℓ_2: all sequ. $\{x_n\}$ s.t. $\sum x_n^2 < \infty$

$$\rho(x,y) = \left(\sum_{n=1}^{\infty} (x_i - y_i)^2\right)^{1/2}$$

HW

$$s(x) = s(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$$

$$s\left(1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\right) = \left(\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\right)$$

$$(a, b, c, d, \ldots) = s(4, a, b, c, d, \ldots)$$

9