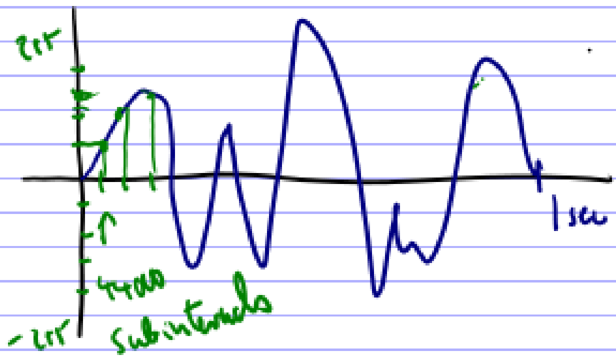
Panel 1

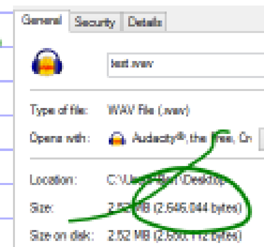
WAV format: e.g. 44.1 KHz, 16 bits



How much info to store Mono

$44\,000 \times 2 = 88\,KB/sec$

⇒ 30 secs of 'A' needs 2.64 MB



How much music fits on CD?

$680\,MB / 88\,000 / 60 / 2 = 65$ mins

Stereo = 2 channels

Actually 75 min to hold Beethoven's 9th Symph

1

Panel 2

If $f(x) = a_0 + \sum_{1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$, $x \in [-\pi, \pi]$

then

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

hard!

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\,dx$$

Compare to Taylor series: $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$,

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$ easy!

2

Panel 3

Jean Baptiste Joseph Fourier

Born	21 March 1768 Auxerre, Burgundy, Kingdom of France (now in Yonne, France)
Died	16 May 1830 (aged 62) Paris, Kingdom of France
Residence	France
Nationality	French
Fields	Mathematician, physicist, and historian
Institutions	École Normale École Polytechnique
Alma mater	École Normale
Doctoral advisor	Joseph Lagrange
Doctoral students	Gustav Dirichlet Giovanni Plana Claude-Louis Navier
Known for	Fourier series Fourier transform

Brook Taylor (1685-1731)

Born	18 August 1685 Edmonton, Middlesex, England
Died	29 December 1731 (aged 46) London, England
Residence	England
Nationality	English
Fields	Mathematician
Institutions	St. John's College, Cambridge
Alma mater	St. John's College, Cambridge
Academic advisors	John Machin and John Keill
Known for	Taylor's theorem Taylor's series

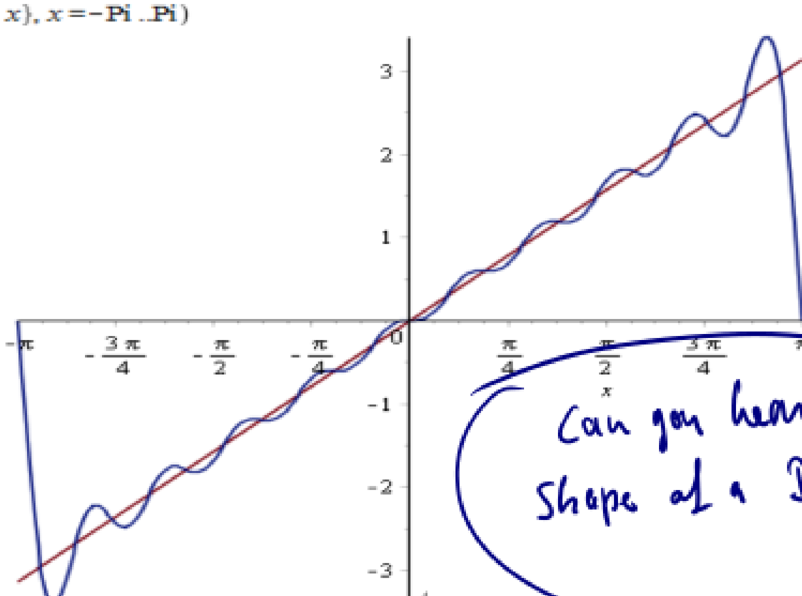
3

Panel 4

$$sum\left(\frac{2\cdot(-1)^{(n+1)}}{n}\cdot\sin(n\cdot x),\ n=1..10\right)$$

$$2\sin(x)-\sin(2x)+\frac{2}{3}\sin(3x)-\frac{1}{2}\sin(4x)+\frac{2}{5}\sin(5x)-\frac{1}{3}\sin(6x)$$
$$+\frac{2}{7}\sin(7x)-\frac{1}{4}\sin(8x)+\frac{2}{9}\sin(9x)-\frac{1}{5}\sin(10x) \approx x\ ,\ [-\pi,\pi]$$

$plot(\{\%, x\}, x=-Pi..Pi)$



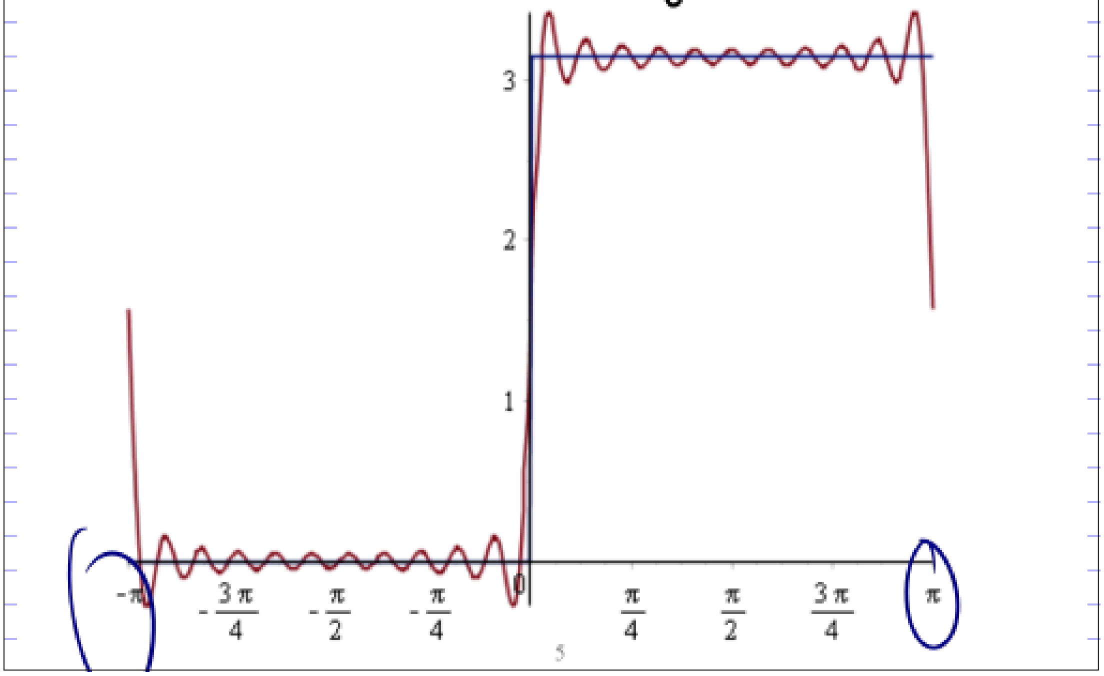
Not quite but close

Can you hear the Shape of a Drum?

4

Panel 5

Fourier Series for $f(x) = \begin{cases} \pi & , x \in (0,\pi] \\ 0 & , x \in [-\pi, 0] \end{cases}$



5

Panel 6

We need to settle which functions have Fourier Series!

Recall Taylor's Thm: f has a Taylor Series iff
$$\lim_{n\to\infty} R_{n+1}(x) = 0 \quad \forall x \in A.$$

http://en.wikipedia.org/wiki/Fourier_series

Def: For all $f : [-\pi,\pi] \to \mathbb{R}$, f Riemann integrable we define
$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

This defines a norm (length) between functions.
We write $\|f\|_2 = \langle f, f\rangle = \int_{-\pi}^{\pi} f^2(x)\,dx$ mean-square norm

6

Panel 7

Def. A sequence $f_n : [-\pi,\pi] \to \mathbb{R}$ of R-int'ble functions is mean-square convergent to f if

$\|f_n - f\| \to 0$ as $n \to \infty$ or $f_n \xrightarrow{L_2} f$

almost everywhere

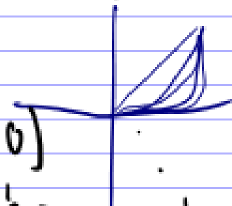
Def. A sequence f_n of functions converges to f a.e. if $f_n \to f$ pointwise except at $x \in E$ and $\mu(E) = 0$

Ex: $f_n(x) = x^n$, $x \in [0,1]$. $f \equiv 0$ $\quad \tilde{f} = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$

Does $f_n \to f$? No, $f(1) \neq 1$

$f_n \to f$ a.e.? yes ($\mu(\{1\}) = 0$)

$\|f_n - f\|_2 \to 0$? yes $\Rightarrow \int_0^1 (f_n - 0)^2\,dx = \int_0^1 x^{2n}\,dx = \frac{1}{2n+1} \to 0$



Panel 8

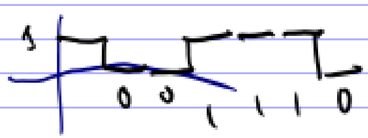
Thm: If $f_n \rightrightarrows f$ (conv. uniformly) then $f_n \xrightarrow{L_2} f$

If $f_n \xrightarrow{L_2} f$ then $f_n \to f$ a.e.

Thm: If f is R-int'ble and f_n is the Fourier series of f then $f_n \xrightarrow{L_2} f$

(thus, $f_n \to f$ a.e.)

Thm: If f is cont on $[-\pi,\pi]$, and f' exists piecewise with f' cont, then Fourier series $f_n \rightrightarrows f$ uniformly!



Panel 9

Need to get rid of annoying interval $[-\pi, \pi]$.

if f is defined on $[-L, L]$ then

$f(\frac{L}{\pi} x)$ is defined on $[-\pi, \pi]$

Thm: f is defined on $[-L, L]$. If f has Fourier Series,

then $f \sim a_0 + \sum_{n=1}^{\infty} a_n \cos(n \frac{\pi x}{L}) + b_n \sin(n \frac{\pi x}{L})$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx \qquad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(n \frac{\pi x}{L})\,dx$$

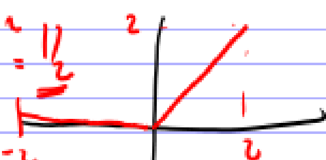
$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \cdot \cos(n \frac{\pi}{L} x)\,dx$$

9

Panel 10

Ex: $f(x) = \begin{cases} 0 & -2 \le x < 0 \\ x & 0 \le x \le 2 \end{cases}$ Fourier Series?

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx = \frac{1}{4} \int_0^2 x\,dx = \frac{1}{8} x^2 \Big|_0^2 = \frac{1}{2}$$



$$a_n = \frac{1}{2} \int_0^2 x \cos(n \frac{\pi x}{2})\,dx = \frac{1}{2}\left[x \frac{2}{\pi n} \sin(n \frac{\pi x}{2}) \Big|_0^2 - \int_0^2 \frac{2}{\pi n} \sin(n \frac{\pi x}{2}) \right.$$

$u = x \qquad u' = 1$

$v' = \cos(n \frac{\pi x}{2}) \qquad v = \frac{2}{\pi n} \sin(n \frac{\pi x}{2})$

$$+ \frac{1}{2} \frac{2}{\pi n} \frac{2}{n\pi} \cos(n \frac{\pi x}{2}) \Big|_0^2$$

$$= \frac{2}{(n\pi)^2}\left((-1)^n - 1\right), \quad b_n = \frac{2}{\pi n} (-1)^{n+1}.$$

10

Panel 11

$$f(x) = \begin{cases} 0 & -2 \leq x < 0 \\ x & 0 \leq x \leq 2 \end{cases} \quad \text{Fourier Series?}$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{2}{(n\pi)^2} \left((-1)^n - 1\right) \cos\left(n \frac{\pi x}{2}\right) + \frac{2}{n\pi} (-1)^{n+1} \cdot \sin\left(n \frac{\pi x}{2}\right) \right)$$

has both b_n's, a_n's.

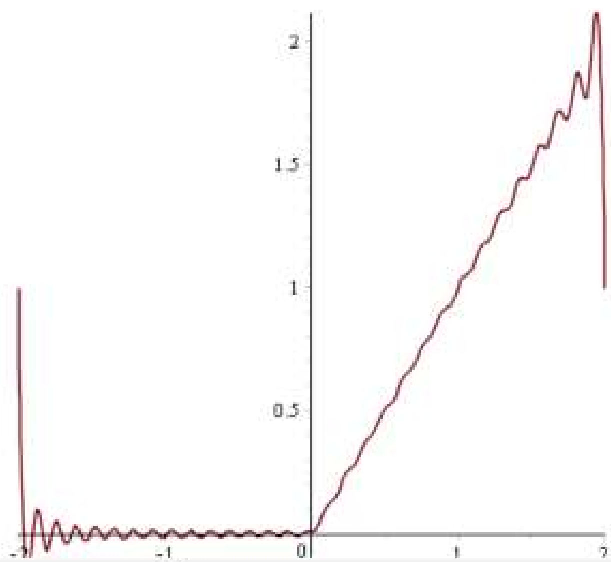
11

Panel 12

$f(x) := \frac{1}{2} + sum\left(\frac{2}{\pi^2 n^2}((-1)^n - 1)\cos\left(\frac{n \cdot \mathrm{Pi} \cdot x}{2}\right) + \frac{2}{n \cdot \mathrm{Pi}}(-1)^{n+1}\sin\left(\frac{n \cdot \mathrm{Pi} \cdot x}{2}\right), n = 1..30 \right)$

$$x \rightarrow \frac{1}{2} + \sum_{n=1}^{30} \left(\frac{2\left((-1)^n - 1\right)\cos\left(\frac{1}{2} n \pi x\right)}{\pi^2 n^2} + \frac{2(-1)^{n+1}\sin\left(\frac{1}{2} n \pi x\right)}{n \pi} \right)$$

$plot(f(x), x = -2..2)$

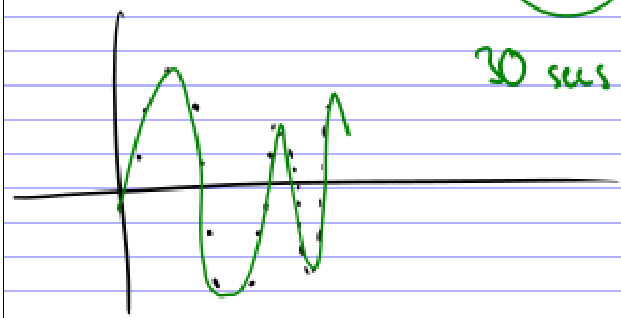


12

Panel 13

MP3 file: sample curve at N points, eg 44000 points and compute Fourier series of f going through these points, store first 1024 Fourier coef. per second.

→ 2049 numbers, 2 bytes per #



30 secs of "A" cont

30 · 2 · 2049 bytes = 122KB

per channel

→ 244KB

13

Panel 14

Need:

2^N points → $\boxed{\text{FFT}}$ → Fourier Coefficients

Fast-Fourier Transform

(Next Line)

14

Panel 15

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gustafso/s2012/2270/web-projects/Guckert-audio-compression-svd-mdct-MP3.pdf

15