Panel 1

$\boxed{\text{Lebesgue}}$ Outer Measure

Def. $\mu^*(A) = \inf\{\sum_i \ell(A_i)\}$ $\{A_i\}$ open intervals covering A

Prop. $\mu^* : P(A) \to [0,\infty)$
$\mu^*(\emptyset) = 0$
if $A \subset B \Rightarrow \mu^*(A) \le \mu^*(B)$
$\mu^*(\cup E_i) \le \sum_i \mu^*(E_i)$

other functions may have these properties $\Rightarrow$ outer measures

Checked: $\mu^*(\text{interval}) = \overline{\text{length of interval}}$
$\mu^*(\text{singleton}) = 0 = \mu^*(\text{countable set})$
$\mu^*(\mathbb{R}) = \infty$

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Panel 2

Theorem: Lebesgue Outer Measure is (countably) subadditive, but not additive

Subadditive: $\mu^*(\cup E_i) \le \sum \mu^*(E_i)$ $(0,2) \cup (1,3)$

Additive $\mu^*(\cup E_i) = \sum \mu^*(E_i)$ E_i disjoint

Find $\overset{\text{some}}{E_i}$ s.t. $\mu^*(\cup E_i) < \sum \mu^*(A_i)$

Really hard:

See:

http://math.stackexchange.com/questions/228866/example-of-strictly-subadditive-lebesgue-outer-measure

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Panel 3

(Lebesgue) Measure: A set E is measurable if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) \quad \forall A$$

and $\mu(E) = \mu^*(E)$.

Corollary: E is msble iff $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$

Proof: $A \subset (A \cap E) \cup (A \cap E^c)$

$$\mu^*(A) \leq \mu^*((A \cap E) \cup (A \cap E^c)) \leq$$

$$\leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

#

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Panel 4

Recall: E is measurable if

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c) \quad \forall A$$

Thm: Any set with outer measure zero is msble.
and $\mu(A) = 0$

Proof: $\mu^*(E) = 0$. $\quad A \cap E \subset E \Rightarrow \mu^*(A \cap E) \leq \mu^*(E) = 0$

$\Rightarrow \mu^*(A \cap E) = 0$

$A \cap E^c \subset A \Rightarrow \underline{\mu^*(A \cap E^c) \leq \mu^*(A)}$

$$\Rightarrow \mu^*(A) \geq \underbrace{\mu^*(A \cap E)}_{0} + \mu^*(A \cap E^c)$$

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Panel 5

Recall: E is measurable if

$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c) \quad \forall A$

Ex: $\emptyset$ and $\mathbb{R}$ are measurable

$\mu^*(\emptyset) = 0 \Rightarrow \mu(\emptyset) = 0$ for $\emptyset$ msble

$\emptyset$

$\mu^*(A) \geq \mu^*(A \cap \mathbb{R}) + \mu^*(A \cap (\mathbb{R})^c)$

$\mu^*(A) + \mu^*(\emptyset) = \mu^*(A)$

$\Rightarrow \mathbb{R}$ is msble $\Rightarrow \mu(\mathbb{R}) = \infty$

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Panel 6

Recall: E is measurable if

$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c) \quad \forall A$

Ex: If E is msble then E^c is msble

$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$

$\mu^*(A) \geq \mu^*(A \cap E^c) + \mu^*(A \cap (E^c)^c) =$

$\mu^*(A \cap E^c) + \mu(A \cap E)$

$\Rightarrow E^c$ is msble

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Panel 7

Recall: E is measurable if

$$\mu^*(A) \ge \mu^*(A \cap E) + \mu^*(A \cap E^c) \quad \forall A$$

Thm: If A, B are msble then $A \cup B$ and $A \cap B$ are msble.

Assume $A \cup B$ is msble. Show $A \cap B$ is

A msble $\Rightarrow$ A^c is msble

B msble $\Rightarrow$ B^c is msble $\Rightarrow A^c \cup B^c = (A \cap B)^c$

$\Rightarrow ((A \cap B)^c)^c$ is msble

$\Rightarrow A \cap B$ is msble.

Panel 8

Need: If E, F msble $\Rightarrow E \cup F$ is msble.

[Proof] (1) $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$ (E msble)

$\mu^*(A) = \mu^*(A \cap F) + \mu^*(A \cap F^c)$ (F msble)

$A \cap (E \cup F) = (A \cap E) \cup (A \cap E^c \cap F)$

$\mu^*(A \cap (E \cup F)) \le \mu^*(A \cap E) + \mu^*(A \cap E^c \cap F)$

$\mu^*(A \cap E^c) = \mu^*(A \cap E^c \cap F) + \mu^*(A \cap E^c \cap F^c)$

$= \mu^*(A \cap E^c \cap F) + \mu^*(A \cap (E \cup F)^c)$

$\mu^*(A) = \underbrace{\mu^*(A \cap E) + \mu^*(A \cap E^c \cap F)} + \mu^*(A \cap (E \cup F)^c)$

$\ge \mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^c)$

Panel 9

If A, B are msble then $A \cup B$ is msble.

Panel 10

Recall: E is measurable if

$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c) \quad \forall A$

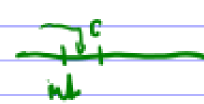
Thm: The intervals (a, ∞) are msble $\forall a \in \mathbb{R}$

Proof: Need $\mu^*(A) \geq \mu^*(A \cap (a, \infty)) + \mu^*(A \cap (-\infty, a])$

Assume $\mu^*(A) < \infty$. Take any $\varepsilon > 0$. Then

$\exists$ collection of open intervals A_n cover A s.t.

$\sum \ell(A_i) \leq \mu^*(A) + \varepsilon$



Take $J_n = A_n \cap (a, \infty)$, $K_n = A_n \cap (-\infty, a]$

Panel 11

$J_n = A_n \cap (a,\infty)$ and $K_n = A_n \cap (-\infty, a]$

J_n, K_n are intervals because A_n are

① $\Rightarrow \ell(J_n) = \mu^*(J_n) \qquad \ell(K_n) = \mu^*(K_n)$

$\ell(J_n) + \ell(K_n) = \ell(A_n)$

② $J_n \subset A_n \Rightarrow \sum \ell(J_n) \leq \sum \ell(A_n)$

$K_n \subset A_n \Rightarrow \sum \ell(K_n) \leq \sum \ell(A_n)$

③ $(A \cap (a,\infty)) \subset \cup J_n \Rightarrow \overbrace{\mu^*(A \cap (a,\infty))}^{*} \leq \mu^*(\cup J_n) \leq \sum \ell(J_n)$ ✓

$(A \cap (-\infty,a]) \subset \cup K_n \Rightarrow \overbrace{\mu^*(A \cap (-\infty,a])}^{**} \leq \mu^*(\cup K_n) \leq \sum \ell(K_n)$ ✓

Finally: $* + ** \leq \sum \ell(J_n) + \sum \ell(K_n) = \sum \ell(J_n) + \ell(K_n)$

$\leq \sum \ell(A_n) \leq \mu^*(A) + \varepsilon$ q.e.d.

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Panel 12

Properties of Lebesgue Measure

(1) All intervals are msble and their msre is length

(2) Any open and any closed set is msble

(3) Union and intersections of msble sets are msble

(4) If $A = \cup A_n$ is msble then $\mu(\underset{n}{\cup} A) \leq \sum \mu(A_n)$

(5) If $A = \cup A_n$ and A_n are disjoint and msble then

$\mu(\underset{n}{\cup} A) = \sum \mu(A_n)$

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Panel 13

Borel sets: Smallest σ-alg. containing all open sets

$\Rightarrow$ open sets, closed sets, unions, intersections, complements are Borel sets

There are Borel sets that are not unions / int. of open / closed sets.

There are sets that are not Borel sets!

All Borel sets are msble

But there are sets that are not measurable!

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Panel 14

$$(1,2] = \bigcap_{n=1}^{\infty} (1, 2+\tfrac{1}{n})$$

Cantor set is Borel set!

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