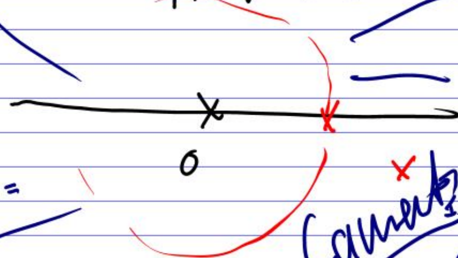
Panel 1

Write $\frac{1}{1-z}$ as a "Power" series:

$$\frac{1}{1-z} = 1+z+z^2+\dots = \sum_{n=0}^{\infty} z^n \quad , |z|<1$$



$$\frac{1}{z\left(\frac{1}{z}-1\right)} = \frac{1}{-z\left(1-\left(\frac{1}{z}\right)\right)} =$$

Laurent Series

$$= -\frac{1}{z}\left(\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right) = -\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots\right)$$

$\left|\frac{1}{z}\right|<1 \Rightarrow |z|>1$ $\quad = -\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\dots = -\sum_{n=1}^{\infty}\left(\frac{1}{z}\right)^n$

Panel 2

Laurent Theorem: If f is analytic in a ring-shaped domain $R_1 < |z-z_0| < R$ then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

where $$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{-n+1}}\,dz$$

$$b_2 = \frac{1}{2\pi i}\int f(z)(z-z_0)^1\,dz$$

Panel 3

Find: Laurent series for $f(z) = e^{1/z}$ centered at $z_0 = 0$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Rightarrow e^{1/z} = \sum_{n=0}^{\infty} (1/z)^n / n! \quad \forall |z| > 0$$

Find Taylor series for $f(z) = 5z^2 + 3z + 1$ centered at 0

$f(z) = a_0 + a_1 z + a_2 z^2 + \dots, \quad a_n = 0, \ n > 2$ $\boxed{a_n = \frac{f^{(n)}(0)}{n!}}$

$= 1 + 3z + \frac{10}{2!} z^2$

this is it

Laurent series $g(z) = \frac{1}{(z-i)^2}$ centered at $z_0 = i$

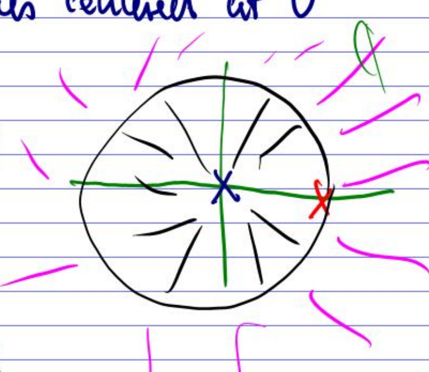
$$g(z) = a_0 + a_1(z-i) + a_2(z-i)^2 + \dots + b_1(z-i)^{-1} + b_2(z-i)^{-2}$$

Panel 4

Ex: Take $f(z) = \frac{1}{2-z}$. Find series centered at 0

a) that converges for $|z| < 2$

$$\frac{1}{2(1-\frac{z}{2})} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n \quad \left|\frac{z}{2}\right| < 1$$



b) that converges for $|z| > 2$

$$\frac{1}{2 - \boxed{z}} = \frac{1}{-z\left(1 - \frac{2}{z}\right)} =$$

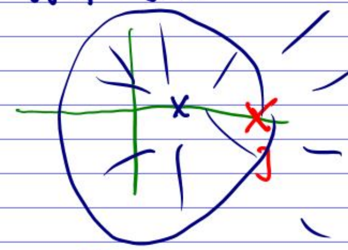
$$= \frac{1}{-z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n, \quad \forall \left|\frac{2}{z}\right| < 1 \iff |z| > 2$$

Panel 5

Ex: $f(z) = \frac{1}{3-z}$. Series centered at $z_0=1$ s.t.

a) convergent for $|z-1|<2$

$$\frac{1}{3-z} = \frac{1}{2-(z-1)} = \frac{1}{2\left(1-\left(\frac{z-1}{2}\right)\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}(z-1)^n$$



b) convergent for $2<|z-1|$

$$\frac{1}{3-z} = \frac{1}{2-(z-1)} = \frac{1}{-(z-1)\left(-\frac{2}{z-1}+1\right)} = \frac{1}{-(z-1)}\sum_{n=0}^{\infty}\left(\frac{2}{z-1}\right)^n = -\sum_{n=0}^{\infty}\frac{2^n}{(z-1)^{n+1}}$$

Panel 6

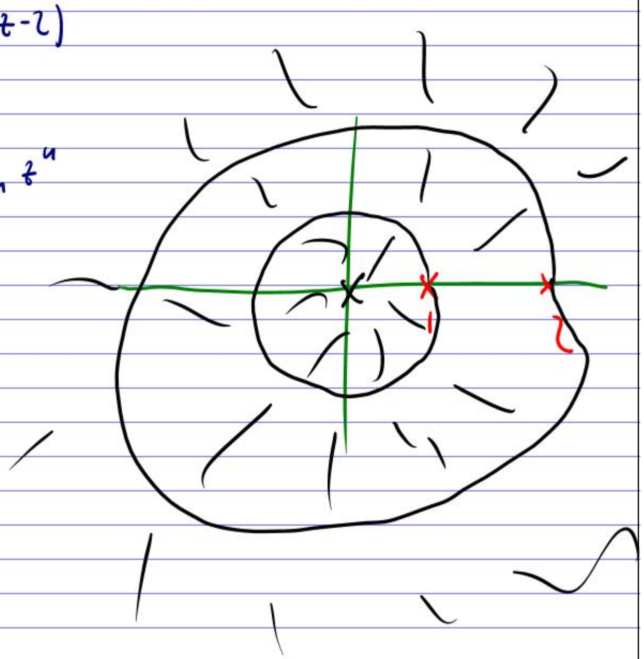
Consider $f(z) = \frac{-1}{(z-1)(z-2)}$

Write $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$

$|z|<1$

$1<|z|<2$

$|z|>2$



Panel 7

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2} \qquad |z| < 1$$

$$= -\frac{1}{1-z} + \frac{1}{2-z} \qquad \frac{1}{2-z} = \frac{1}{2\left(1-\frac{z}{2}\right)}$$

$$\forall |z|<1 \quad = -\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \forall |z|<2$$

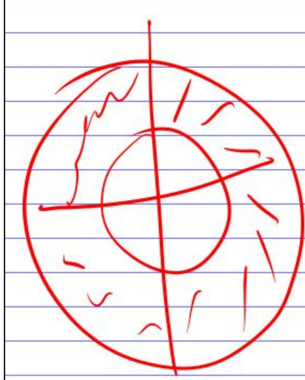
$$= \sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}} - 1\right) z^n \qquad \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$\forall |z| < 1$$

Panel 8

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2} \qquad 1 < |z| < 2$$

$$= \frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{2\left(1-\frac{z}{2}\right)}$$



$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n$$

$$1 < |z| \qquad |z| < 2$$

Panel 9

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2} \qquad \forall |z| > 2$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \frac{1}{z\left(1 - \frac{2}{z}\right)} =$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \frac{1}{z} \sum \frac{2^n}{z^n} =$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} =$$

$$= \sum_{n=0}^{\infty} \left(1 - 2^n\right) \frac{1}{z^{n+1}}$$

Panel 10

$$f(z) = \frac{1}{(z-1)(z-2)}$$, Compute $f(1.5)$

using a series.

$$f(1.5) = \sum_{n=0}^{\infty} \frac{1}{(1.5)^{n+1}} + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} (1.5)^n =$$

$$= \frac{1}{-0.5^2} = -\frac{1}{0.25} = -4$$

Panel 11

Complex HW

Find the Laurent series centered at $z_0 = 0$, and list specifically the value of the coefficient $a_{-1} =$ ___

a) $f(z) = z^4 \sin(1/z)$ b) $g(z) = \frac{1}{z^4} \sin(z)$

The function $f(z) = \frac{1}{3-z}$ has two series expansions, one for $|z| < 3$ and another for $|z| > 3$. Find them.

$f(z) = \frac{-2}{(z-1)(z-3)}$ has 3 series expansions. Find all three series and their domain of convergence.

How many series expansions centered at $z_0 = 0$ does $f(z) = \frac{1}{e^z - 1}$ have, and where do they converge?

Panel 12

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

$$\sin(1/z) = \left(\frac{1}{z} - \frac{1}{z^3 3!} + \frac{1}{z^5 5!} + \dots \right)$$

$$\Rightarrow a_{-1} = 1$$

$$z^4 \sin\left(\frac{1}{z}\right) = z^3 - z/3! + \frac{1}{z \cdot 5!} - \frac{1}{z^3 7!}$$

$$\Rightarrow a_{-1} = 1/5!$$

$$\frac{1}{z^4} \sin(z) = \frac{1}{z^3} - \frac{1}{z\, 3!} + \frac{z}{5!} - \frac{z^3}{7!} + \dots$$

$$\Rightarrow a_{-1} = -1/3!$$

Panel 13

If $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$, then a_{-1} is called

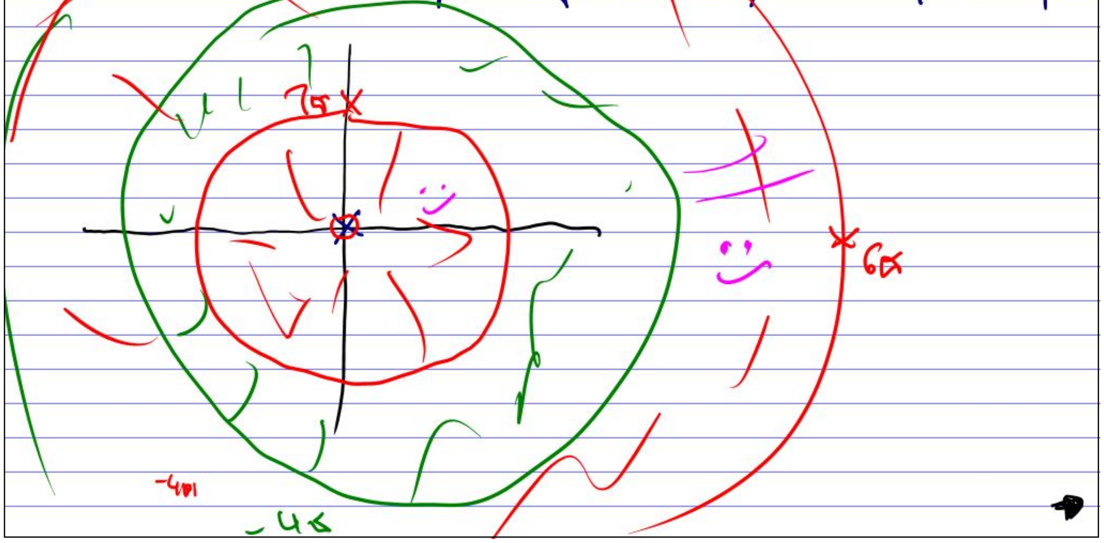
Residue of f centered at z_0

$\underline{\operatorname{Res}(f, z_0)}$

Panel 14

(4) How many series expansions centered at $z_0 = 0$ does $f(z) = \frac{1}{e^z - 1}$ have, and where do they converge?

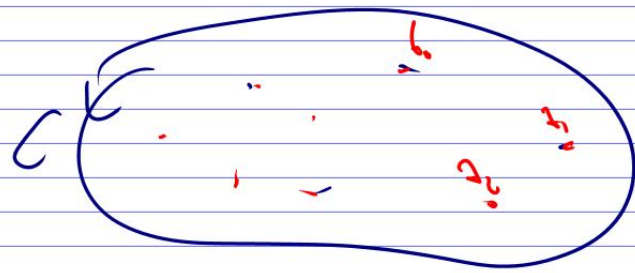
$f(z)$ is not analytic if $z = 0,\ 2\pi i,\ 4\pi i, \ldots$



Panel 15

Residue Theorem: If f is analytic in D except for finitely many points $z_0, z_1, z_2, \ldots, z_n$. Then

$$\int_C f(z)\,dz = 2\pi i \sum_{i=0}^{n} \mathrm{Res}(f, z_i)$$



Panel 16

(1) $$\int_C \frac{z^2+1}{(z-1)(z-3)}\,dz, \quad C: |z|<2$$

(2) $\frac{1}{(z-5)(z-1)}$ find series that converges for $|z|<1$

$1<|z|<5$

$5<|z|$