Panel 1

We defined special functions:

e^z

$\cos(z)$

$\sin(z)$

$\log(z) =$

$\mathrm{Log}(z) =$

Panel 2

If $z(t) = x(t) + iy(t)$, $t \in [a,b]$ is a path in $\mathbb{C}$:

$$\int_a^b z(t)\,dt =$$

If $z(t)$, $t \in [a,b]$ is the parametrization of a path γ in $\mathbb{C}$ and $f: \mathbb{C} \to \mathbb{C}$ a complex function,

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

Panel 3

More about Paths:

① Find a line segment from A to B, $A, B \in \mathbb{C}$

$$l(t) = A + (B-A)\cdot t, \quad t \in [0,1]$$

② Find circle of radius R and center A

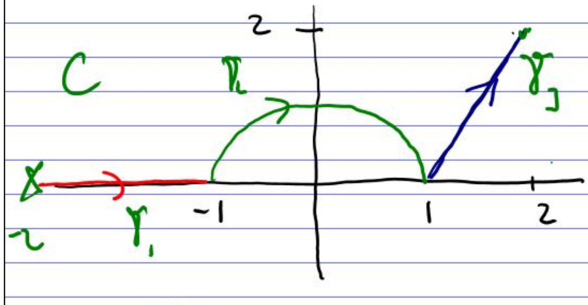
$$r(t) = Re^{it} + A, \quad t \in [0, 2\pi]$$

$$R(\cos t + i \sin t) + a_1 + i a_2 \qquad (x-a_1)^2 + (y-a_2)^2 = R^2$$

$$x = R\cos(t) + a_1, \quad y = R\sin(t) + a_2$$

Panel 4

Ex: Parametrize the following curve



piecewise smooth,

$$C = \gamma_1 + \gamma_2 + \gamma_3$$

γ_1: $l(t) = -2 + (-1-(-2))t = -2 + t, \quad t \in [0,1]$

γ_2: $r(t) = e^{it}, \quad t \in [\pi, 0]$

γ_3: $l(t) = 1 + (2+2i-1)t = 1 + (1+2i)t, \quad t \in [0,1]$

Panel 5

Ex: Let γ be the straight line from $z_0 = 0$ to $z_1 = 1+i$.

Compute $\displaystyle\int_\gamma z^2\, dz$

$$\gamma : z(t) = 0 + (1+i-0)\,t = (1+i)\,t \quad , t \in [0,1]$$

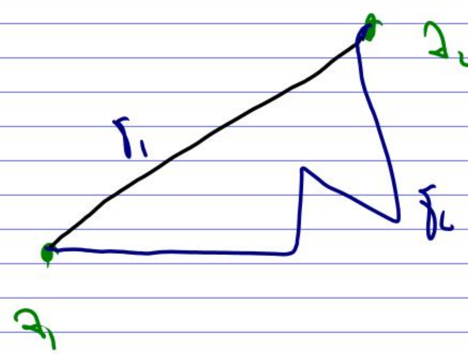
$$\int_0^1 \big((1+i)\,t\big)^2 (1+i)\, dt = \int (1+i)^2 t^2 (1+i)\, dt =$$

$$= (1+i)\int_0^1 2i\, t^2\, dt = 2i(1+i)\frac{1}{3}$$

Panel 6

Natural questions:

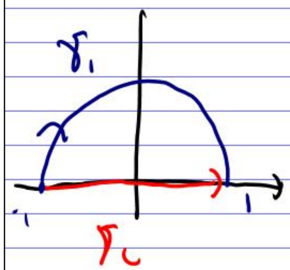
$$\int_{z_1}^{z_2} f(z)\, dz \qquad Q: \text{ is}$$



$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz$$

Panel 7

Ex: Evaluate $\int_{\gamma_1} z^2\,dz$ and $\int_{\gamma_2} z^2\,dz$



γ_1 and γ_2 have same start and endpoint.

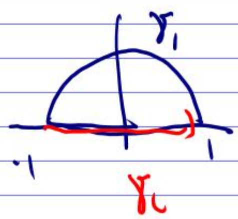
$\gamma_1: z(t) = e^{it},\ t \in [\pi, 0]$

$$\int_{\gamma_1} z^2\,dz = \int_\pi^0 (e^{it})^2\, i e^{it}\,dt = i\int_\pi^0 e^{i3t}\,dt = i \cdot \frac{1}{3i} e^{i3t}\Big|_\pi^0 = \frac{1}{3}\left(1 - e^{i3\pi}\right) = \frac{2}{3}$$

$\gamma_2: z(t) = t,\ t \in [-1,1]:\ \int_{\gamma_2} z^2\,dz = \int_{-1}^1 t^2\,dt = \frac{1}{3} t^3\Big|_{-1}^1 = \frac{2}{3}$ //

Panel 8

Ex: Evaluate $\int_{\gamma_1} \bar{z}\,dz$ and $\int_{\gamma_2} \bar{z}\,dz$



$\gamma_1: z(t) = e^{it},\ t \in [\pi, 0]$

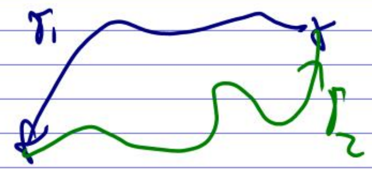
$$\int_\pi^0 e^{-it}\, i e^{it}\,dt = i(0-\pi) = \underline{-\pi i}$$

$$\int_{\gamma_2} \bar{z}\,dt = \int_{-1}^1 t\,dt = \underline{0}$$

$z(t) = t,\ t \in [-1,1]$

Panel 9

Bad news: For two paths from z_1 to z_2 it may happen that $\int_{\gamma_1} f(z)\,dz \neq \int_{\gamma_2} f(z)\,dz$



Later

Conjecture: If f is analytic, then

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz,$$ γ_1, γ_2 have same beginning and end!

Panel 10

Compute $\int_{|z|=1} z^{10}\,dz$, $z(t) = e^{it}$, $t \in [0, 2\pi]$

$$= \int_0^{2\pi} \left(e^{it}\right)^{10} i e^{it}\,dt$$

$$= i \int_0^{2\pi} e^{i11t}\,dt =$$

$$= i \frac{1}{11i} e^{i11t} \Big|_0^{2\pi} =$$

$$= e^{i22\pi} - 1$$

$$= 1 - 1 = 0$$

Panel 11

$$\int_{|z|=1} z^n \, dz = \int_0^{2\pi} \left(e^{it}\right)^n \cdot i e^{it} \, dt =$$

$$= i \int_0^{2\pi} e^{i(n+1)t} \, dt =$$

$$= \cancel{i} \frac{1}{(n+1)\cancel{i}} e^{i(n+1)t} \Big|_0^{2\pi}$$

$$= \frac{1}{(n+1)} \Big(\underbrace{e^{i2(n+1)\pi}}_{=1} - 1 \Big) = 0$$

Panel 12

$$\int_{|z|=1} z^n \, dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases}$$

$$\int_{|z|=1} \frac{1}{z} \, dz = \int_0^{2\pi} \frac{1}{\cancel{e^{it}}} \cdot i \cancel{e^{it}} \, dt = i \int_0^{2\pi} dt = \underline{2\pi i}$$

Panel 13

<u>Theorem.</u> Suppose $f(z)$ is continuous in D. Then the following are equivalent:

(a) f has antiderivative F, i.e. $F'(z) = f$

(b) If γ_1 is any path from z_1 to z_2, then

$$\int_\gamma f(z)\,dz = F(z_2) - F(z_1)$$

(c) $\displaystyle\int_\gamma f(z)\,dz = 0$ for every <u>closed</u> curve γ

Panel 14

<u>Ex:</u> $\displaystyle\int_\gamma z^2\,dz$, γ from 2 to i

Old way: $z(t) = 2 + (i-2)t$, $t \in [0,1]$

$$\int_0^1 \left(2+(i-2)t\right)^2 (i-2)\,dt = \int_0^1 \left(4 + 4(i-2)t + (i-2)^2 t^2\right)(i-2)\,dt$$

$$= (i-2)\left(4 + 4(i-2)\tfrac{1}{2} + (i-2)^2 \tfrac{1}{3}\right) = -\frac{8}{3} - \frac{i}{3}$$

<u>New way:</u> $f(z) = z^2$ has antiderivative $F(z) = \frac{1}{3} z^3$

$$\Rightarrow \int_\gamma z^2\,dz = \int_2^i z^2\,dz = \frac{1}{3} z^3 \Big|_2^i = \frac{1}{3}\left(i^3 - 2^3\right) = -\frac{i}{3} - \frac{8}{3}$$