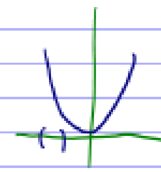
Panel 1

Last Time

Max. Modulus Principle: f is analytic inside a domain D, then f has no max. in D



Series: $\sum_{n=0}^{\infty} z_n = z_0+z_1+z_2+\dots$

$S_n = z_0+z_1+\dots+z_n$ If $\{S_n\}$ converges as a sequence

Def then $\sum_{j=0}^{\infty} z_j$ converges

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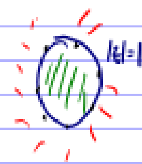
Panel 2

Thm: The Geometric Series $\sum_{n=0}^{\infty} z^n$ converges if $|z|<1$ and diverges if $|z|>1$.



Moreover $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$, if $|z|<1$

Ex: $\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n = 1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3+\dots+\dots = \frac{1}{1-\left(\frac{3}{4}\right)} = \underline{4}$

$\sum_{j=3}^{\infty}\left(\frac{1}{5}\right)^n = \left(\frac{1}{5}\right)^3+\left(\frac{1}{5}\right)^4+\left(\frac{1}{5}\right)^5+\left(\frac{1}{5}\right)^6+\dots = \frac{1}{1-1/5}-\left(\frac{1}{5}\right)^2-\left(\frac{1}{5}\right)-1$

$= \left(\frac{1}{5}\right)^3\left(1+\frac{1}{5}+\left(\frac{1}{5}\right)^2+\left(\frac{1}{5}\right)^3+\dots\right) = \left(\frac{1}{5}\right)^3\frac{1}{1-1/5} = \frac{1}{5^3}\cdot\frac{5}{4} = \frac{1}{100}$

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Panel 3

Def. A series $\sum_{n=0}^{\infty} z_n$ converges absolutely if

$$\sum_{n=0}^{\infty} |z_n| \text{ converges}$$

Thm: If $\sum_{n=0}^{\infty} z_n$ converges absolutely then

$\sum_{n=0}^{\infty} z_n$ converges (because $z_n \leq |z_n|$)

Thm: If $\sum_{n=0}^{\infty} z_n$ converges then $\lim_{n\to\infty} z_n = 0$ Thm

Divergence

$S_n = z_0 + z_1 + z_2 + \dots + z_n \Rightarrow S_n - S_{n-1} = z_n$

$0 = L - L = \lim_{n\to\infty} \left(S_n - S_{n-1}\right) = \lim z_n$, L is $\sum_{n=0}^{\infty} z_n$ limit

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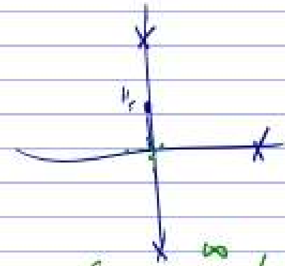
Panel 4

$\sum_{n=0}^{\infty} \left(\frac{i}{5}\right)^n$ converges because $\left|\frac{i}{5}\right| = \frac{1}{5} < 1$ (Geometric Series)

$\sum_{n=0}^{\infty} \left(1 - \frac{i}{n}\right)$ diverges



Check $\lim_{n\to\infty} \left(\frac{i}{5}\right)^n = 0$

$\lim_{n\to\infty} \left(1 - \frac{i}{n}\right) = 1$

Note: Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

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Panel 5

Def: A series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n =$$

is called a power series centered at z_0.

Ex: $\frac{1}{1-z} = f(z) = \sum_{n=0}^{\infty} z^n$ is a power series centered at zero

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Panel 6

Taylor's Theorem: If f is analytic in the disk $|z - z_0| < R$ then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad |z - z_0| < R$$

where $a_n = \frac{f^{(n)}(z_0)}{n!}$. In other words,

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \dots$$

$$= f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \frac{f'''(z_0)}{3!}(z - z_0)^3 + \dots$$

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Panel 7

Taylor Series centred at z_0 (also called McLaurin series) for

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} z^n \qquad \text{Recall: } a_n = \frac{f^{(n)}(z_0)}{n!}$$

$a_0 = f(0) = 1$

$a_1 = \frac{f'(0)}{1!} = 1 \qquad f = (1-z)^{-1}$

$a_2 = \frac{f''(0)}{2!} = \frac{2!}{2!} = 1 \qquad f' = +(1-z)^{-2}$

$a_3 = \frac{f'''(0)}{3!} = \frac{3!}{3!} = 1 \qquad f'' = +2(1-z)^{-3}$

$\vdots \qquad f''' = 3!(1-z)^{-4}$

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Panel 8

Ex: $f(z) = e^z = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$

$a_0 = f(0) = 1$

$a_1 = \frac{f'(0)}{1!} = \frac{1}{1!}$

Corollary: $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

$= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$

$a_2 = \frac{f''(0)}{2!} = \frac{1}{2!}$

$e^2 = \left(1 + 2 + \frac{4}{2!} + \frac{8}{3!} + \frac{16}{4!} + \dots\right) = \sum_{n=0}^{\infty} \frac{2^n}{n!}$

$a_3 = \frac{f'''(0)}{3!} = \frac{1}{3!}$

$\vdots$

Corollary: $\lim_{n\to\infty} \frac{2^n}{n!} = 0$

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Panel 9

$$\frac{d}{dz} e^z = \frac{d}{dz}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots\right)$$

$$= 0 + 1 + 2\frac{z}{2!} + 3\frac{z^2}{3!} + 4\frac{z^3}{4!} + \dots$$

$$= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots = e^z$$

$$\int_0^1 e^{z^2} dz = \int_0^1 \sum_{n=0}^{\infty} \frac{(z^2)^n}{n!} dz = \int_0^1 1 + z^2 + \frac{z^4}{2!} + \frac{z^6}{3!} + \frac{z^8}{4!} + \dots dz$$

$$1 + \frac{1}{3} + \frac{1}{2!}\cdot\frac{1}{5} + \frac{1}{3!}\cdot\frac{1}{7} + \dots$$

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Panel 10

Find Taylor series of $\underline{z^3 e^z}$ centered at $z_0 = 0$

$a_0 = f(0) = 0$

$a_1 = f'(0) = 3z^2 e^z + z^3 e^z \Big|_{z=0} = 0$

$a_2 = \frac{1}{2!} f''(0) = \underline{\qquad\qquad}$ pain!

$a_3 =$

$$z^3 e^z = z^3\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right) = z^3 + \frac{z^4}{1!} + \frac{z^5}{2!} + \frac{z^6}{3!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{z^{n+3}}{n!}$$

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