

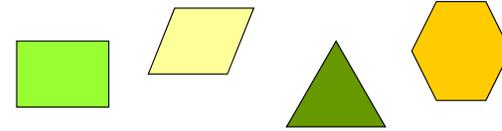
Chapter 5

Linear Inequalities and Linear Programming

Section 3

Linear Programming in Two Dimensions: A Geometric Approach

Linear Programming in Two Dimensions: A Geometric Approach



In this section, we will explore applications which utilize the graph of a system of linear inequalities.

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A Familiar Example

We have seen this problem before. An extra condition will be added to make the example more interesting.

Suppose a manufacturer makes two types of skis: a trick ski and a slalom ski. Each trick ski requires 8 hours of design work and 4 hours of finishing. Each slalom ski requires 8 hours of design and 12 hours of finishing. Furthermore, the total number of hours allocated for design work is 160 and the total available hours for finishing work is 180. Finally, the number of trick skis produced must be less than or equal to 15. How many trick skis and how many slalom skis can be made under these conditions?

Now, here is the twist: Suppose the profit on each trick ski is \$5 and the profit for each slalom ski is \$10. How many each of each type of ski should the manufacturer produce to earn the greatest profit?

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Linear Programming Problem

This is an example of a **linear programming problem**. Every linear programming problem has two components:

1. A linear **objective function** is to be maximized or minimized. In our case the objective function is Profit = $5x + 10y$ (5 dollars profit for each trick ski manufactured and \$10 for every slalom ski produced).
2. A collection of linear inequalities that must be satisfied simultaneously. These are called the **constraints** of the problem because these inequalities give limitations on the values of x and y . In our case, the linear inequalities are the constraints.

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Constraints

$x \geq 0$
 $y \geq 0$
 $x \leq 15$
 $8x + 8y \leq 160$
 $4x + 12y \leq 180$

The number of trick skis must be less than or equal to 15

Finishing constraint: 4 hours for each trick ski and 12 hours for each slalom ski.

x and y have to be positive

Design constraint: 8 hours to design each trick ski and 8 hours to design each slalom ski.

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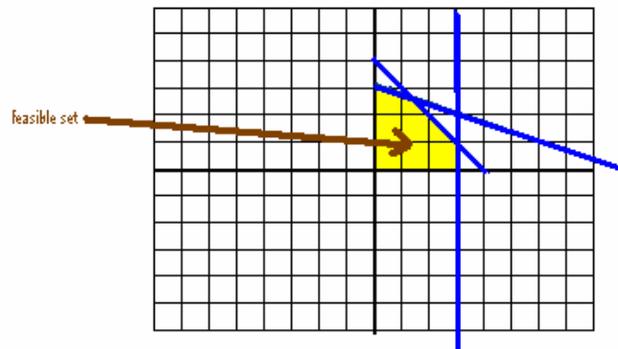
Linear Programming Problem (continued)

3. The **feasible set** is the set of all points that are possible solutions. In this case, we want to determine the value(s) of x , the number of trick skis and y , the number of slalom skis that will yield the maximum profit. Only certain points are eligible. Those are the points within the common region of intersection of the graphs of the constraining inequalities. Let's return to the graph of the system of linear inequalities. Notice that the feasible set is the yellow shaded region.

Our task is to maximize the profit function $P = 5x + 10y$ by producing x trick skis and y slalom skis, but use only values of x and y that are within the yellow region graphed in the next slide.

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The Feasible Set



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Maximizing the Profit

The profit is given by $P = 5x + 10y$.

The equation $k = 5x + 10y$ represents a line with slope $(-1/2)$. For each point (x, y) on this line, the profit equals k . This is called an **constant-profit line**. As the profit k increases, the line shifts upward by the amount of increase while remaining parallel to all other constant-profit lines.

What we are attempting to do is to find the largest value of k possible.

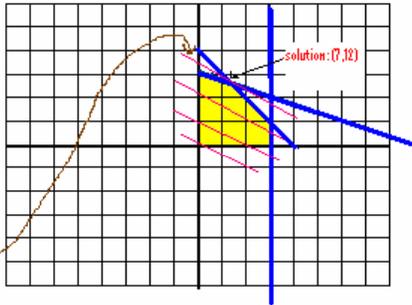
The graph on the next slide shows a few isoprofit lines. The maximum value of profit occurs at a corner point - a point of intersection of two lines.

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Constant-profit Lines

The exact point of intersection of the two lines is (7.5,12.5). Since x and y must be whole numbers, we round the answer down to (7,12).

Various profit levels are indicated by the pink lines. Each line has a slope of $-1/2$ the slope of the line $5x + 10y = P$, where P is a constant. The profit level that is greatest and yet still includes one or more points of the feasible set is indicated.



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Maximizing the Profit (continued)

The maximum value of the profit function in this example was at the corner point, (7.5, 12.5), but since we cannot produce a fraction of a ski, we will round down to (7, 12). We cannot exceed the constraints, so we can't round up.

Thus, the manufacturer should produce 7 trick skis and 12 slalom skis to achieve maximum profit.

What is the maximum profit?

$$P = 5x + 10y$$

$$P = 5(7) + 10(12) = 35 + 120 = 155.$$

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General Result

- If a linear programming problem has a solution, it is located at a **corner point** of the set of feasible solutions. If a linear programming problem has more than one solution, at **least** one of them is located at a corner point of the set of feasible solutions.
- If the set of feasible solutions is bounded, as in our example, the solution of the linear programming problem will exist. *Bounded* means that the region can be enclosed in a circle.
- If the set of feasible solutions is not bounded, then the solution may or may not exist. Use the graph to determine whether a solution exists or not.

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Constructing a Model for a Linear Programming Problems

1. Introduce decision variables.
2. Summarize relevant material in table form, relating columns to the decision variables, if possible.
3. Determine the objective and write a linear objective function.
4. Write problem constraints using linear equations and/or inequalities.
5. Write non-negative constraints.

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Geometric Method for Solving Linear Programming Problems

1. Graph the feasible region. Then, if an optimal solution exists, find the coordinates of each corner point.
2. Construct a **corner point table** listing the value of the objective function at each corner point.
3. Determine the optimal solution(s) from the table in step 2.
4. For an applied problem, interpret the optimal solution(s) in terms of the original problem.

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Example 1

Maximize the quantity $z = x + 2y$ subject to the constraints $x + y \geq 1$, $x \geq 0$, $y \geq 0$.

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Example 1 (no solution)

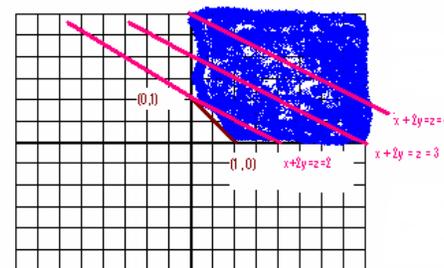
Maximize the quantity $z = x + 2y$ subject to the constraints $x + y \geq 1$, $x \geq 0$, $y \geq 0$.

1. The objective function is $z = x + 2y$, which is to be maximized.
2. Graph the constraints: (see next slide)
3. Determine the feasible set (see next slide)
4. Determine the corner points of the feasible set. There are two corner points from our graph: $(1,0)$ and $(0,1)$
5. Determine the value of the objective function at each vertex. At $(1, 0)$, $z = (1) + 2(0) = 1$; at $(0, 1)$, $z = 0 + 2(1) = 2$.

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Example 1 (continued)

The pink lines are the graphs of $z = x + 2y$ for $z = 2, 3$ and 4 . We can see from the graph there is no feasible point that makes z largest. The region is unbounded.



We conclude that this linear programming problem has no solution.

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Example 2



A manufacturing plant makes two types of boats, a two-person boat and a four-person boat. Each two-person boat requires 0.9 labor-hours from the cutting department and 0.8 labor-hours from the assembly department. Each four-person boat requires 1.8 labor-hours from the cutting department and 1.2 labor-hours from the assembly department. The maximum labor-hours available per month in the cutting department and the assembly department are 864 and 672, respectively. The company makes a profit of \$25 on each two-person boat and \$40 on each four-person boat. How many boats of each kind should the company produce in order to maximize profit?

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Example 2 (unique solution)

1. Write the objective function.
2. Write the problem constraints and the nonnegative constraints.
3. Graph the feasible region. Find the corner points.
4. Test the corner points in the objective function to find the maximum profit.

Answers: 1. Objective Function

If x is the number of two-person boats and y is the number of four-person boats, and the company makes a profit of \$25 on each two-person boat and \$40 on each four-person boat, the objective function is $P = 25x + 40y$.

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Example 2 (continued)

2. Constraints

Since each 2-person boat requires 0.9 labor hours from the cutting department and each 4-person boat requires 1.8 hour of cutting, and the maximum hours available in the cutting department is 864, we have $0.9x + 1.8y \leq 864$

Since each 2-person boat requires 0.8 labor-hours from assembly and each 4-person requires 1.2 hour from assembly, and the maximum hours available in assembly is 672, we have $0.8x + 1.2y \leq 672$.

Also, $x \geq 0$ and $y \geq 0$.

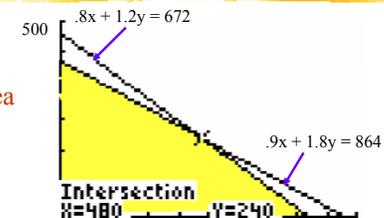
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Example 2 (continued)

3. Feasible region

The yellow shaded area is the feasible region.

Corner points are $(0, 0)$, $(0, 480)$, $(480, 240)$, and $(840, 0)$



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Example 2 (continued)

4. Maximum Profit

Corner point	Value of $P = 25x + 40y$
(0,0)	0
(0,480)	$40(480) = \$19,200$
(480,240)	$25(480) + 40(240) = \$21,600$ This is max. profit
(840,0)	$25(840) = \$21,000$

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Example 3

Maximize $z = 4x + 2y$ subject to

$$2x + y \leq 20$$

$$10x + y \geq 36$$

$$2x + 5y \geq 36$$

$$x, y \geq 0$$

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Example 3 (multiple solution)

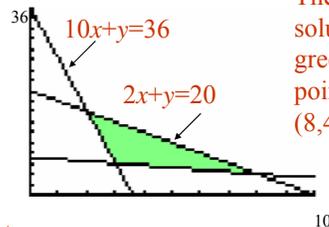
Maximize $z = 4x + 2y$ subject to

$$2x + y \leq 20$$

$$10x + y \geq 36$$

$$2x + 5y \geq 36$$

$$x, y \geq 0$$



The feasible solution is shaded green. The corner points are (2,16), (8,4) and (3,6).

Test corner points:

(2,16)	$z = 4(2) + 2(16) = 40$	} These are both optimal.
(8,4)	$z = 4(8) + 2(4) = 40$	
(3,6)	$z = 4(3) + 2(6) = 20$	

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Example 3 (continued)

The solution to example 3 is a **multiple optimal solution**. In general, if two corner points are both optimal solutions to a linear programming problem, then any point on the line segment joining them is also an optimal solution. Thus any point on the line $2x + y = 20$, where $2 \leq x \leq 8$, such as (3, 14), would be an optimal solution.



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