Math 3626 - Practice Exam

Note Title 11/12/2015

(1) $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$.

$\Rightarrow \det(A) = 1 \cdot 3 - 2 \cdot 0 = 3$

Eigenvalues: $\det(A - \lambda \, Id) = 0$

$\Leftrightarrow \quad (1-\lambda)(3-\lambda) - 0 = 0$

$\Rightarrow \quad \lambda = 1, 3$

Eigenvector: $A\vec{v} = \lambda \vec{v}$ or $(A - \lambda Id) \cdot \vec{v} = 0$

$\lambda = 1: \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$\Leftrightarrow \quad 2v_2 = 0$ so $v_2 = 0$, v_1 arbitrary

$\Rightarrow \quad \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda = 3: \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$\Rightarrow v_1 = v_2$ so that

$\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Inverse: $\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array}\right) \to \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1/3 \end{pmatrix} \to$

$\begin{pmatrix} 1 & 0 & 1 & -2/3 \\ 0 & 1 & 0 & 1/3 \end{pmatrix} \Rightarrow A^{-1} = \begin{pmatrix} 1 & -2/3 \\ 0 & 1/3 \end{pmatrix}$

check: $A \cdot A^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

(2) $A = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ -3 & 2 & 1 \end{pmatrix}$ is symmetric so all Eigenvalues are real!

$\det(A) = 1 \cdot \det\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - 0 \cdot \# - 3 \det\begin{pmatrix} 0 & -3 \\ 1 & 2 \end{pmatrix}$

$= 1(-2) - 3 \cdot 3 = -11$

Eigenvalues: $\lambda_1 = 1 + \sqrt{13}$

$\lambda_2 = 1 - \sqrt{13}$

$\lambda_3 = 1$

Eigenvectors: $\vec{v}_1 = \langle -\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}, 1 \rangle$

$\vec{v}_2 = \langle \frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}}, 1 \rangle$

$\vec{v}_3 = \langle 2, 3, 0 \rangle$

Inverse:

$$\begin{pmatrix} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ -3 & 2 & 1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & -8 & 3 & 0 & 1 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -12 & 3 & -2 & 1 \end{pmatrix} \to \begin{pmatrix} 4 & 0 & 0 & 1 & 2 & -1 \\ 0 & 6 & 0 & 3 & 4 & 1 \\ 0 & 0 & -12 & 3 & -2 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & 1/4 & 1/2 & -1/4 \\ 0 & 1 & 0 & 1/2 & 2/3 & 1/6 \\ 0 & 0 & 1 & -1/4 & 1/6 & -1/12 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1/4 & 1/2 & -1/4 \\ 1/2 & 2/3 & 1/6 \\ -1/4 & 1/6 & -1/12 \end{pmatrix} \qquad A \cdot A^{-1} = Id \ \checkmark$$

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$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & -1 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow 0 = 1 \quad \underline{\underline{\text{No solution}}}$$

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$$\begin{aligned} x + 2y &= 4 \\ 3y &= 6 \end{aligned} \qquad \Rightarrow \begin{aligned} x &= 0 \\ &\underline{\underline{y = 2}} \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & -3 & 6 \\ 0 & 1 & 2 & 10 \\ -3 & 2 & 1 & 18 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -3 & 6 \\ 0 & 1 & 2 & 10 \\ 0 & \cancel{2}\,1 & -\cancel{2}\,1 & \cancel{24}\ {-14} \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & -3 & 6 \\ 0 & 1 & 2 & 10 \\ 0 & 0 & 3 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 10 \\ 0 & 0 & 3 & -4 \end{pmatrix}$$

$\Rightarrow x = 2, \quad z = -\frac{4}{3}, \quad y = 10 + \frac{8}{3} = \frac{38}{3}$

(5)

1	3	0	1	0	0	0	10
4	1	3	0	1	0	0	8
1	0	6	0	0	1	0	12
-2	-4	-1	0	0	0	1	0

1/3	1	0	1/3	0	0	0	10/3
11/3	0	3	-1/3	1	0	0	14/3
1	0	6	0	0	1	0	12
-2/3	0	-1	4/3	0	0	1	$\frac{40}{3}$

(6) First one is done:

$x = 0, \quad y = 7/4, \quad z = \frac{33}{4}, \quad P = \frac{111}{4}$ is max

Second one: pivot in col 1, row 2

(7) Use our "doSimplex" program

```
doSimplex[{{10,5},{12,1},{3,2},{-4,-2}}, {50,48,18,0
```

$x = 19/5, \quad y = 12/5, \quad P = 20$

```
doSimplex[{{4,-3,1},{1,1,1},{2,1,-1},{-2,3,-4}},
{10,8,12,0}]
```

$x=0,\ y=0,\ z=8,\ P=32$

8) $A = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & 4 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 0 \end{pmatrix}$

Dual Problem is:

$$\begin{pmatrix} 1 & 4 & 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 & 0 & 3 \\ -1 & -3 & 0 & 0 & 1 & 0 \end{pmatrix}$$

solve using doSimplex but check "tableau" for answer

```
doSimplex[{{ 1,4},{1,2},{-1,-3}},{1,3,0}]
```

$$\begin{array}{cccccc} s_1 & s_2 & x & y & & \\ \end{array}$$
$$\begin{pmatrix} 1 & 4 & 1 & 0 & 0 & 1 \\ 0 & -2 & -1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$x=1,\ y=0,\ P=1$ <u>is min</u>

Summary:

- Setup coefficient matrix A
- Find A^T, insert identity matrix, change signs in last row
- Pivot as usual
- When done, <u>last row</u> <u>contains solution</u>

(10) a) $3n^8 + 2n^2 - 10$ is $\underline{\underline{O(n^8)}}$

b) $n^2 + 2^n + \log_2(n)$ is $\underline{\underline{O(2^n)}}$

c) $\sqrt{n} + \sqrt[3]{n^2} + \ln(n) =$

$n^{1/2} + n^{2/3} + \ln(n)$ is $\underline{\underline{O(n^{2/3})}}$

(11) Adding matrices: $O(n)$

Mult matrices. $O(n^2)$

$\det(M)$ is $O(n!)$ (usual algorithm)

(12) a) $y' = 6y^2x$, $y(6) = -1/80$

$$\int \frac{dy}{y^2} = \int 6x\,dx \iff -\frac{1}{y} = 3x^2 + C$$

$80 = 3 \cdot 36 + C \Rightarrow \underline{\underline{C = -28}}$

$$\Rightarrow y = -\frac{1}{3x^2 - 28}$$

$\lim_{x \to \infty} y(x) = 0$

b) $y'\sqrt{1+x^2} = xy^3$, $y(0) = -1$

$$\int \frac{dy}{y^3} = \int \frac{x}{\sqrt{1+x^2}}\,dx$$

$$-2y^{-2} = \sqrt{1+x^2} + c$$

$$-2 = c$$

$$\Rightarrow y = \sqrt{\frac{1}{\sqrt{1+x^2}-2}}$$

(17) $$y''' - 5y'' + 9y = t\cos(2t)$$

$$x_1 = y$$

$$x_1' = x_2 = y'$$

$$x_2' = x_3 = y''$$

$$x_3' = 5y'' - 9y + t\cos(2t)$$

$\Rightarrow X' = A \cdot X$ where $X = (x_1, x_2, x_3)$ and

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & 0 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ t\cos(2t) \end{pmatrix}$$

check. $X' = A \cdot X + G$

$$x_1' = x_2$$
$$x_2' = x_3$$
$$x_3' = -9x_1 + 5x_3$$

$y^{(4)} = -3y''' - 5y' + 6y + 11$

$x_1 = y \quad , x_2 = y' , \quad x_3 = y'' , X_4 = y'''$

$x_4' = -3x_4 - 5x_2 + 6x_1 + 11$

$$X' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 6 & -5 & 0 & -3 \end{pmatrix} X + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 11 \end{pmatrix}$$

No, this conversion is <u>not</u> unique!

(14) a) Solve $\underbrace{X' = \begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix}}_{A} X \quad , x(0) = (10, 0)$

Eigenvalues $[A]$: $\lambda_1 = 6$

$\lambda_2 = -1$

Eigenvectors $[A]$: $v_1 = (3, 4)$

2 Real EVs

$v_2 = (-1, 1)$

Solutions: $X(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} =$

$$c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

or: $x_1 = c_1\, 3e^{6t} - c_2 e^{-t}$

$x_2 = c_1\, 4e^{6t} + c_2 e^{-t}$

$X(0) = \langle 10, 6 \rangle$

$\Rightarrow \begin{matrix} 3c_1 - c_2 = 10 \\ 4c_1 + c_2 = 6 \end{matrix} \Rightarrow 7c_1 = 16 \Rightarrow \underline{\underline{c_1 = \frac{16}{7}}}$

$c_2 =$

(2)

Solve $X' = AX$; $A = \begin{pmatrix} -1 & -6 \\ 3 & 5 \end{pmatrix}$

$\lambda_1 = 2+3i \quad , \lambda_2 = 2-3i$

$v_1 = \langle -1+i, 1 \rangle, \quad v_2 = \langle -1-i, 1 \rangle$

2 complex conjugate EV

Let $\lambda = Re(\lambda_1), \mu = Im(\lambda_1)$

$A = Re(\vec{v_1}), \quad B = Im(\vec{v_1})$

Solution

$\Rightarrow X = c_1 e^{\lambda t}\left(A\cos(\mu t) - B\sin(\mu t) \right) + c_2 e^{\lambda t}\left(B\cos(\mu t) + A\sin(\mu t) \right)$

So: $\lambda = Re(2+3i) = 2, \mu = Im(2+3i) = 3$

$A = \langle -1, 1 \rangle \qquad B = \langle 1, 0 \rangle$

$$\to x(t) = c_1 e^{2t}\left(\begin{pmatrix}-1\\1\end{pmatrix}\cos(3t) - \begin{pmatrix}1\\0\end{pmatrix}\sin(3t)\right) +$$

$$c_2 e^{2t}\left(\begin{pmatrix}1\\0\end{pmatrix}\cos(3t) + \begin{pmatrix}-1\\1\end{pmatrix}\sin(3t)\right)$$

or:

$$\underline{x_1(t) =} \; c_1 e^{2t}\left(-\cos(3t) - \sin(3t)\right) +$$

$$c_2 e^{2t}\left(\cos(3t) - \sin(3t)\right)$$

$$\underline{x_2(t) =} \; c_1 e^{2t}\cos(3t) + c_2 e^{2t}\sin(3t)$$

Check that $x_1' = -x_1 - 6x_2$ and

$x_2' = 3x_1 + 5x_2$!

(C) $X' = AX$, $A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & 5 \\ 0 & 5 & 1 \end{pmatrix}$

$\lambda_1 = \frac{3}{2}(1+\sqrt{13})$, $\lambda_2 = \frac{3}{2}(1-\sqrt{13})$, $\lambda_3 = 1$

$v_1 = \langle -\frac{2}{5}, \frac{1}{10}((+3)\sqrt{13}), 1 \rangle$

$v_2 = \langle -\frac{2}{5}, \frac{1}{10}(1-3\sqrt{13}), 1 \rangle$

$v_3 = \langle 5, 0, 2 \rangle$

Solution: (for all eigenvalues real)

$$X(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} + c_3 v_3 e^{\lambda_3 t}$$

Double-check: $X' = A \cdot X$ ✓

(15) $y'' + 2y' = 7y \qquad y(0) = 1$

$\Rightarrow y'' = -2y' + 7y$

Let: $x_1 = y$

$x_2 = y'$

$\Rightarrow \quad x_2' = 7x_1 - 2x_2$ or equiv.

$X' = AX$, where $A = \begin{pmatrix} 0 & 1 \\ 7 & -2 \end{pmatrix}$

$\lambda_1 = -1 - 2\sqrt{2}$

$\lambda_2 = -1 + 2\sqrt{2}$

$v_1 = \langle \frac{1}{7}(-1 - 2\sqrt{2}), 1 \rangle$

$v_2 = \langle \frac{1}{7}(-1 + 2\sqrt{2}), 1 \rangle$

Solution

$$X = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

so that $y = x_1 = c_1 \frac{1}{7}\left(-1-2\sqrt{2}\right) e^{(-1-2\sqrt{2})t} + c_2 \frac{1}{7}\left(-1+2\sqrt{2}\right) e^{(-1+2\sqrt{2})t}$

check answer: $y'' + 2y' = 7y$

using Mathematica:

```
y[t_] = c1 1/7 (-1 - 2 Sqrt[2]) Exp[(-1 - 2 Sqrt[2]) t] + c2 1/7 (-1 + 2 Sqrt[2]) Exp[(-1 + 2 Sqrt[2]) t]
```

$$\frac{1}{7}\left(-1-2\sqrt{2}\right) c1\, e^{\left(-1-2\sqrt{2}\right)t} + \frac{1}{7}\left(-1+2\sqrt{2}\right) c2\, e^{\left(-1+2\sqrt{2}\right)t}$$

```
y''[t] + 2 y'[t] == 7 y[t]
```

$$\frac{1}{7}\left(-1-2\sqrt{2}\right)^3 c1\, e^{\left(-1-2\sqrt{2}\right)t} + \frac{1}{7}\left(-1+2\sqrt{2}\right)^3 c2\, e^{\left(-1+2\sqrt{2}\right)t} + 2\left(\frac{1}{7}\left(-1-2\sqrt{2}\right)^2 c1\, e^{\left(-1-2\sqrt{2}\right)t} + \frac{1}{7}\left(-1+2\sqrt{2}\right)^2 c2\, e^{\left(-1+2\sqrt{2}\right)t}\right) == 7\left(\frac{1}{7}\left(-1-2\sqrt{2}\right) c1\, e^{\left(-1-2\sqrt{2}\right)t} + \frac{1}{7}\left(-1+2\sqrt{2}\right) c2\, e^{\left(-1+2\sqrt{2}\right)t}\right)$$

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Simplify[%]
True
```

so it does indeed check out!