**Completeness Theorem. A real sequence is Cauchy if and only if it converges to some limit *a*.**:

First, assume that the sequence converges to some limit *a*. Take any *http://www.mathcs.org/analysis/reals/symbols/epsi.gif > 0*. There exists an integer *N* such that if *j > N* then *| aj - a | < http://www.mathcs.org/analysis/reals/symbols/epsi.gif /2*. Hence:

*| aj - ak | http://www.mathcs.org/analysis/reals/symbols/le.gif | aj - a | + | a - ak| < 2 http://www.mathcs.org/analysis/reals/symbols/epsi.gif / 2 = http://www.mathcs.org/analysis/reals/symbols/epsi.gif*

if *j, k > N*. Thus, the sequence is Cauchy.

Second, assume that the sequence is Cauchy (this direction is much harder). Define the set

***S****= {x http://www.mathcs.org/analysis/reals/symbols/element.gif****R****: x < aj for all j except for finitely many}*

Since the sequence is bounded (by part one of the theorem), say by a constant *M*, we know that every term in the sequence is bigger than *-M*. Therefore *-M* is contained in ***S***. Also, every term of the sequence is smaller than *M*, so that ***S*** is bounded by *M*. Hence, ***S*** is a non-empty, bounded subset of the real numbers, and by the least upper bound property it has a well-defined, unique least upper bound. Let

*a = sup(****S****)*

We will now show that this *a* is indeed the limit of the sequence. Take any *http://www.mathcs.org/analysis/reals/symbols/epsi.gif > 0* , and choose an integer *N > 0* such that

*| aj - ak | < http://www.mathcs.org/analysis/reals/symbols/epsi.gif / 2*

if *j, k > N*. In particular, we have:

*| aj - aN + 1 | < http://www.mathcs.org/analysis/reals/symbols/epsi.gif / 2*

if *j > N*, or equivalently

*- http://www.mathcs.org/analysis/reals/symbols/epsi.gif / 2 < aj - aN + 1 < http://www.mathcs.org/analysis/reals/symbols/epsi.gif / 2*

Hence we have:

*aj > aN + 1 - http://www.mathcs.org/analysis/reals/symbols/epsi.gif / 2*

for *j > N*. Thus, *aN + 1 - http://www.mathcs.org/analysis/reals/symbols/epsi.gif / 2* is in the set ***S***, and we have that

*a http://www.mathcs.org/analysis/reals/symbols/ge.gif aN + 1 - http://www.mathcs.org/analysis/reals/symbols/epsi.gif / 2*

It also follows that

*aj < aN + 1 + http://www.mathcs.org/analysis/reals/symbols/epsi.gif / 2*

for *j > N*. Thus, *aN + 1 + http://www.mathcs.org/analysis/reals/symbols/epsi.gif / 2* is not in the set ***S***, and therefore

*a http://www.mathcs.org/analysis/reals/symbols/le.gif aN + 1 + http://www.mathcs.org/analysis/reals/symbols/epsi.gif / 2*

But now, combining the last several line, we have that:

*|a - aN + 1 | < http://www.mathcs.org/analysis/reals/symbols/epsi.gif / 2*

and together with the above that results in the following:

*| a - aj | < |a - aN + 1 | + | aN + 1 - aj | < 2 http://www.mathcs.org/analysis/reals/symbols/epsi.gif / 2 = http://www.mathcs.org/analysis/reals/symbols/epsi.gif*

for any *j > N*.

**Bolzano-Weierstrass: Every bounded sequence of real numbers contains a convergent subsequence**

Since the sequence is bounded, there exists a number *M* such that *| aj | < M* for all *j*. Then:

either *[-M, 0]* or *[0, M]* contains infinitely many elements of the sequence

Say that *[0, M]* does. Choose one of them, and call it http://www.mathcs.org/analysis/reals/symbols/a_j_1.gif

either *[0, M/2]* or *[M/2, M]* contains infinitely many elements of the (original) sequence.

Say it is *[0, M/2]*. Choose one of them, and call it http://www.mathcs.org/analysis/reals/symbols/a_j_2.gif

either *[0, M/4]* or *[M/4, M/2]* contains infinitely many elements of the (original) sequence

This time, say it is *[M/4, M/2]*. Pick one of them and call it http://www.mathcs.org/analysis/reals/symbols/a_j_3.gif

Keep on going in this way, halving each interval from the previous step at the next step, and choosing one element from that new interval. Here is what we get:

* *|http://www.mathcs.org/analysis/reals/symbols/a_j_1.gif - http://www.mathcs.org/analysis/reals/symbols/a_j_2.gif| < M*, because both are in *[0, M]*
* *|http://www.mathcs.org/analysis/reals/symbols/a_j_2.gif - http://www.mathcs.org/analysis/reals/symbols/a_j_3.gif| < M / 2*, because both are in *[0, M/2]*
* *|http://www.mathcs.org/analysis/reals/symbols/a_j_3.gif - http://www.mathcs.org/analysis/reals/symbols/a_j_4.gif| < M / 4*, because both are in *[M/2, M/4]*

and in general, we see that

*|http://www.mathcs.org/analysis/reals/symbols/a_j_k.gif - http://www.mathcs.org/analysis/reals/symbols/a_j_k&1.gif| < M / 2k-1*

because both are in an interval of length *M / 2k-1*. So, this proves that consecutive elements of this subsequence are close together. That is not enough, however, to say that the sequence is Cauchy, since for that not only consecutive elements must be close together, but all elements must get close to each other eventually.

So: take any *http://www.mathcs.org/analysis/reals/symbols/epsi.gif > 0*, and pick an integer *N* such that ???...??? (This trick is often used: first, do some calculation, then decide what the best choice for *N* should be. Right now, we have no way of knowing a good choice). Pretending, however, that we knew this choice of *N*, we continue the proof. For any *k, m > N* (with *m > k*) we have:

*http://www.mathcs.org/analysis/reals/numseq/graphics/bolz1.gif   
      http://www.mathcs.org/analysis/reals/numseq/graphics/bolz2.gif   
      http://www.mathcs.org/analysis/reals/numseq/graphics/bolz3.gif   
      http://www.mathcs.org/analysis/reals/numseq/graphics/bolz4.gif   
      http://www.mathcs.org/analysis/reals/numseq/graphics/bolz5.gif   
      http://www.mathcs.org/analysis/reals/numseq/graphics/bolz6.gif*

Now we can see the choice for *N*: we want to make is so large, such that whenever *k, m > N*, the difference between the members of the subsequence is less than the prescribed http://www.mathcs.org/analysis/reals/symbols/epsi.gif. What is therefore the right choice for *N* to finish the proof ?

**Def.: lim sup and lim inf**

Let http://www.mathcs.org/analysis/reals/symbols/%7Ba_j%7D_j.gif be a sequence of real numbers. Define

*Aj = inf{aj , aj + 1 , aj + 2 , ...}*

and let *c = lim (Aj)*. Then *c* is called the **limit inferior** of the sequence http://www.mathcs.org/analysis/reals/symbols/%7Ba_j%7D_j.gif.

Let http://www.mathcs.org/analysis/reals/symbols/%7Ba_j%7D_j.gif be a sequence of real numbers. Define

*Bj = sup{aj , aj + 1 , aj + 2 , ...}*

and let *c = lim (Bj)*. Then *c* is called the **limit superior** of the sequence http://www.mathcs.org/analysis/reals/symbols/%7Ba_j%7D_j.gif.

In short, we have:

1. *lim inf(aj) = lim(Aj)* , where *Aj = inf{aj , aj + 1 , aj + 2 , ...}*
2. *lim sup(aj) = lim(Bj)* , where *Bj = sup{aj , aj + 1 , aj + 2 , ...}*

**What is *inf*, *sup*, *lim inf* and *lim sup* for http://www.mathcs.org/analysis/reals/numseq/graphics/%7B(1)j%7D.gif?**

Clearly, the infimum of the sequence is *-1*, and the supremum is *+1*. To find *lim inf* and *lim sup*, we will first find the sequence of numbers *Aj* and *Bj* mentioned in the definition.

Let's find the numbers *Aj = inf{aj, aj + 1, aj + 2, ...}* for the sequence *{-1, 1, -1, 1, ...}*.

* *A1 = inf{-1, 1, -1, 1, ...} = -1*
* *A2= inf{1, -1, 1, -1, ... } = -1*

and so on. Therefore, it is clear that

*lim inf http://www.mathcs.org/analysis/reals/numseq/graphics/%7B(1)j%7D.gif= -1*

Similarly, we find the numbers *Bj = sup{aj, aj + 1, Aj + 2, ...}*:

* *B1 = sup{-1, 1, -1, 1, ...} = 1*
* *B2 = sup{1, -1, 1, -1, ...} = 1*

and so on. Therefore, it is clear that

*lim suphttp://www.mathcs.org/analysis/reals/numseq/graphics/%7B(1)j%7D.gif= 1*

**What is *inf*, *sup*, *lim inf* and *lim sup* for http://www.mathcs.org/analysis/reals/numseq/graphics/%7B1_fr_j%7D.gif ?**

Since this sequence is *{1, 1/2, 1/3, 1/4, ...}* the infimum is zero, while the supremum is 1. As for *lim inf* and *lim sup*, we find first the sequence of numbers *Aj* and *Bj* mentioned in the definition.

* *A1 = inf{1, 1/2, 1/3, 1/4, ...} = 0*
* *A2 = inf{1/2, 1/3, 1/4, 1/5, ...} = 0*
* *A3 = inf(1/3, 1/4, 1/5, 1/6, ...} = 0*

and so on. Therefore, it is clear that

*lim inf http://www.mathcs.org/analysis/reals/numseq/graphics/%7B1_fr_j%7D.gif= 0*

Similarly, we find the numbers *Bj= sup{aj, aj + 1, aj + 2, ...}:*

* *B1 = sup{1, 1/2, 1/3, 1/4, ...} = 1*
* *B2 = sup{1/2, 1/3, 1/4, 1/5, ...} = 1/2*
* *B3 = sup(1/3, 1/4, 1/5, 1/6, ...} = 1/3*

and so on. Therefore, it is clear that

*lim sup http://www.mathcs.org/analysis/reals/numseq/graphics/%7B1_fr_j%7D.gif= 0*

**What is *inf*, *sup*, *lim inf* and *lim sup* for http://www.mathcs.org/analysis/reals/numseq/graphics/%7Bm1j%7D.gif**

This sequence is *{-1, 2, -3, 4, -5, 6, -7, ...}*. You can quickly check, by looking at the definition of *lim inf* and *lim sup* and working out the numbers *Aj* and *Bj* that:

* *inf { (-1) j j } = - http://www.mathcs.org/analysis/reals/symbols/infinity.gif*
* *lim inf { (-1) j j } = - http://www.mathcs.org/analysis/reals/symbols/infinity.gif*
* *sup{ (-1) j j } = http://www.mathcs.org/analysis/reals/symbols/infinity.gif*
* *lim sup{ (-1) j j } = http://www.mathcs.org/analysis/reals/symbols/infinity.gif*

Note that while *lim sup* and *lim inf* are not real numbers, they are uniquely defined as plus or minus infinity. The limit of the original sequence, on the other hand, does not exist at all.

Hence, there is a difference between a limit not existing, and a limit that approaches infinity. In the latter ense, *lim inf* and *lim sup* will always exist, which is their most useful property.

***lim sup* and *lim inf* always exist (possibly infinite) for any sequence of real numbers.**

**Proof:**

The sequence *Aj = inf{aj , aj + 1 , aj + 2 , ...}* is monotone increasing. Hence, *lim inf* exists (possibly positive infinity).

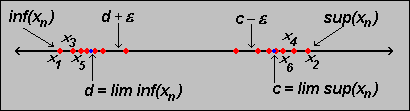
The sequence *Bj = sup{aj , aj + 1 , aj + 2 , ...}* is monotone decreasing. Hence, *lim sup* exists (possibly negative infinity).

Here we have to allow for a limit to be positive or negative infinity, which is different from saying that a limit does not exist.

**Let http://www.mathcs.org/analysis/reals/symbols/%7Ba_j%7D_j.gif be an arbitrary sequence, and let *c = lim sup aj* and *d = lim inf aj*. Then**

1. **there is a subsequence converging to *c***
2. **there is a subsequence converging to *d***
3. ***d http://www.mathcs.org/analysis/reals/symbols/le.gif lim inf http://www.mathcs.org/analysis/reals/symbols/a_j_k.gif http://www.mathcs.org/analysis/reals/symbols/le.gif lim sup http://www.mathcs.org/analysis/reals/symbols/a_j_k.gif http://www.mathcs.org/analysis/reals/symbols/le.gif c* for any subsequence *{http://www.mathcs.org/analysis/reals/symbols/a_j_k.gif}***

**If *c* and *d* are both finite, then: given any *http://www.mathcs.org/analysis/reals/symbols/epsi.gif> 0* there are arbitrary large *j* such that *aj > c -http://www.mathcs.org/analysis/reals/symbols/epsi.gif* and arbitrary large *k* such that *ak < d + http://www.mathcs.org/analysis/reals/symbols/epsi.gif*.**

****

Proof - skipped

**If http://www.mathcs.org/analysis/reals/symbols/%7Ba_n%7D_n.gifis the sequence of all rational numbers in the interval [0, 1], enumerated in any way, find the *lim sup* and *lim inf* of that sequence.**

Since the numbers 1 and 0 are itself rational numbers, it is clear that *sup{ an } = 1* and *inf{ an } = 0*  Therefore, we already know that *0 http://www.mathcs.org/analysis/reals/symbols/le.gif lim inf{ an } http://www.mathcs.org/analysis/reals/symbols/le.gif lim sup{ an } http://www.mathcs.org/analysis/reals/symbols/le.gif 1*

To find the *lim sup*, we will construct a subsequence that converges to 1:

* there exists *0 < aj1 < 1* with *1 - aj1 < 1*
* there exists *0 < aj2 < 1* with *1 - aj2 < 1/2* and *aj1 # aj2*
* there exists *0 < aj3 < 1* with *1 - aj3 < 1/3* and *aj3* different from the previous ones
* and so on ...

These numbers exist because the rational numbers in the interval [0, 1] are arbitrarily close to any real number in that interval.

The subsequence *{ ajk }* constructed in the above way converges to 1. We already know that any limit of any convergent subsequence must be less than or equal to 1. Therefore, since the *lim sup* is the greatest limit of any convergent subsequence, we have

*lim sup { an } = 1*

Similarly, we can extract a subsequence *{ ajk }* that converges to 0. We also know that every limit of any convergent subsequence must be greater or equal to zero. Therefore, since the *lim inf* is the smallest possible limit of all convergent subsequence, we have:

*lim inf { an } = 0*

The final statement relates *lim sup* and *lim inf* with our usual concept of limit.

**If a sequence *{aj}* converges then** ***lim sup aj = lim inf aj = lim aj***

**Conversely, if *lim sup aj = lim inf aj* are both finite then *{aj}* converges.**

To see that even simple concepts like *lim inf* and *lim sup* can result in interesting math consider the following *unproven conjecture*:

If *pn* is the n-th prime number, then *lim inf pn+1 - pn = 2* and *lim sup pn+1 - pn = http://www.mathcs.org/analysis/reals/symbols/infinity.gif*

The first equation is a conjecture, not yet proven, called the **twin prime conjecture**. In fact, it is not even known if the*lim inf* is finite. On the other hand, the second equation involving *lim sup* is known to be infinite because of arbitrary spaces between two primes.

**Power Sequence**http://www.mathcs.org/analysis/reals/numseq/graphics/special1.gif

**Exponent Sequence**http://www.mathcs.org/analysis/reals/numseq/graphics/special2.gif

**Root of n Sequence**http://www.mathcs.org/analysis/reals/numseq/graphics/special3.gif

**n-th Root Sequence**http://www.mathcs.org/analysis/reals/numseq/graphics/special4.gif

**Binomial Sequence**http://www.mathcs.org/analysis/reals/numseq/graphics/special5.gif

**Euler's Sequence**http://www.mathcs.org/analysis/reals/numseq/graphics/special6.gif

**Exponential Sequence**http://www.mathcs.org/analysis/reals/numseq/graphics/special7.gif