Panel 1

Real Analysis - Last time

$\mathbb{R}$ is different from $\mathbb{Q}$ because of lub-property:

every non-empty set with upper bound has least upper bound.

$\mathbb{R} = \mathbb{Q} \cup \{\text{stuff to satisfy lub-prop}\}$

Prop: There is no $x \in \mathbb{Q}$ with $x^2 = 2$

There is no $x \in \mathbb{Q}$ with $x^2 = 3$

There is an $x \in \mathbb{R}$ with $x^2 = 2$ !

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Panel 2

Prop: There is an $x \in \mathbb{R}$ s.t. $x^2 = 2$

Proof: Define $S = \{t \in \mathbb{R} : t \geq 0, t^2 \leq 2\}$

and $s = \sup(S)$

$S$ is not empty $\Rightarrow 1 \in S$

$S$ has upper bdd $\Rightarrow 2 \notin S$

$\boxed{s \text{ exists}}$

Know: $1 \leq s \leq 2$

Want: $s^2 = 2$

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Panel 3

$$S = \{ t \in \mathbb{R} : t \geq 0 \text{ and } t^2 \leq 2 \}, \quad s = \sup(S)$$

Assume $s^2 < 2$: define $\varepsilon = \frac{2 - s^2}{2s+1} \left( < \frac{2}{3} \right)$

Know: $0 < \varepsilon < 1$

$$(s+\varepsilon)^2 = s^2 + 2s\varepsilon + \varepsilon^2 < s^2 + 2s\varepsilon + \varepsilon = s^2 + \varepsilon(2s+1) =$$
$$= s^2 + \cancel{(2s+1)}\left(\frac{2 - \cancel{s^2}}{\cancel{2s+1}}\right) = 2$$

$(s+\varepsilon)^2 \leq 2 \Rightarrow s+\varepsilon \in S$

But $s+\varepsilon > s$, so $s$ is no upper bound ↯

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Panel 4

$$S = \{ t \in \mathbb{R} : t \geq 0 \text{ and } t^2 \leq 2 \}, \quad s = \sup(S)$$

Assume $s^2 > 2$: $\varepsilon = \frac{s^2 - 2}{2s} > 0$

$$(s-\varepsilon)^2 = s^2 - 2\varepsilon s + \varepsilon^2 \geq s^2 - 2\varepsilon s = \cancel{s^2} - \cancel{2s}\left(\frac{\cancel{s^2} - 2}{\cancel{2s}}\right) = 2$$

$\Rightarrow (s-\varepsilon)^2 \geq 2$

But then $s-\varepsilon$ is upper bound for $S$

$s - \varepsilon < s$, so $s$ can't be least upper bound ↯

$\Rightarrow s^2 = 2$

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Panel 5

## Sequences

**Def.** A sequence is defined as a function $f: \mathbb{N} \to \mathbb{R}$. Thus, a sequence can be written as $f(1) = a_1$, $f(2) = a_2$, $f(3) = a_3$

We usually write $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$ or $\{a_1, a_2, \ldots\}$

**Q:** Set $S$ of all $f: \mathbb{N} \to \mathbb{N}$. What is $\operatorname{card}(S) = c$ / $S$ is all seqn. with int. coeff.

**Def.** A sequence $\{a_j\}$ converges to a limit $c$ if:

Given any $\varepsilon > 0$ $\exists N$ s.t. whenever $j > N$

$\Rightarrow |a_j - c| < \varepsilon$

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Panel 6

**Ex:** Prove that $\{1/j\}$ converges to $0$

Take any $\varepsilon > 0$. Pick $N$ s.t. $N \cdot \varepsilon > 1$ Archimedian Prop.

If $j \geq N$ $\Rightarrow \frac{1}{j} \leq \frac{1}{N} < \varepsilon$

$|a_j - 0| < \varepsilon$ as long as $j > N$

# q.e.d

**Note** $N$ (almost) always depends on $\varepsilon$

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Panel 7

Ex: Prove that $\{(-1)^j\}$ does not converge

Suppose $a_j \to L$. Then $\exists N$ s.t. $|a_j - L| < \frac{1}{2}$ $(\epsilon)$ if $j \geq N$

$j$ = even: $|1 - L| < \frac{1}{2}$, $j \geq N \iff -\frac{1}{2} < 1 - L < \frac{1}{2}$

$j$ = odd: $|-1 - L| < \frac{1}{2}$, if $j \geq N \iff -\frac{1}{2} < -1 - L < \frac{1}{2}$

$-\frac{1}{2} < 1 - L < \frac{1}{2} \Rightarrow -\frac{3}{2} < -L < -\frac{1}{2} \Rightarrow \frac{3}{2} > L > \frac{1}{2}$

$-\frac{1}{2} < -1 - L < \frac{1}{2} \Rightarrow \frac{1}{2} < -L < \frac{3}{2} \Rightarrow -\frac{3}{2} < L < -\frac{1}{2}$

↯

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Panel 8

Theorem: Suppose $\{a_n\}$ converges. Then
(1) $\{a_n\}$ is bounded
(2) the limit is unique

Proof of (1): Assume $a_n$ conv. Then for any $\epsilon > 0$
$\exists N$ s.t. $|a_j - c| < \epsilon$, if $j \geq N$

Pick $\epsilon = 1$. $\exists N$ s.t. $|a_j - c| < 1$ if $j \geq N$

$|a_j| = |a_j - c + c| \leq |a_j - c| + |c| \leq 1 + |c|$ if $j \geq N$

Pick $M = \max\{|a_1|, |a_2|, \ldots, |a_N|, 1 + |c|\}$

$\Rightarrow |a_j| \leq M \quad \forall j$

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Panel 9

If $a_n$ converges, the limit is unique.

Proof: Assume $a_j \to L_1 \Rightarrow \forall \varepsilon > 0\ \exists N$ s.t. $|a_j - L_1| < \varepsilon/2$

and $a_j \to L_2$ $\forall \varepsilon > 0\ \exists N_2$ s.t. $|a_j - L_2| < \varepsilon/2$

$\Rightarrow |L_1 - L_2| = |L_1 - a_j + a_j - L_2| < |L_1 - a_j| + |a_j - L_2| =$

$= |a_j - L_1| + |a_j - L_2| \leq$

$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon$

Two # $L_1, L_2$ s.t. $|L_1 - L_2| < \varepsilon$ for any $\varepsilon > 0$ !!

$\Rightarrow L_1 = L_2$

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Panel 10

Q: If $\{a_n\}$ converges, then $\{a_n\}$ is bdd.
Converse?

Q: If $a_n \to L_1$ and $b_n \to L_2$, and
$a_n < b_n$ $\forall n$. Is it true that
$L_1 < L_2$ as well?

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