**Major Theorems of Line Integration (so far)**

**Fundamental Theorem**: If a vector field is ***conservative*** with potential function , and is a curve from point A to point B, then

**Path Independence**: If is a ***conservative*** vector field and and are two curves, both starting at point A and ending at point B, then

**Closed Loop**: If is a ***conservative*** vector field and is a ***closed curve***, then

The above theorems only work for *conservative* vector fields (do you recall how to check if a 2D or 3D vector field is conservative?). There is another theorem that applies to any vector field, as long as it is 2 dimensional (we will cover a 3D version later). But (everything has its price) it only applies to vector fields over closed curves. Still, it is a very useful and interesting theorem; it is:

**Green’s Theorem:** If is a 2D vector field and is a closed curve, positively orientated, with interior . Then

This is a “big deal” theorem because it lets you compare “apples with oranges”, i.e. things that you normally cannot compare. It can be very useful in finding the work integral, so that **whenever** you see an integral of a **2D vector field over a closed curve**, you should always **check** if **Green’s theorem** would simplify the problem. Here are a few examples:

* Evaluate , where C is the ellipse

It is a 2D vector field over a closed curve, so we (automatically) apply Green’s theorem: and so that and . Thus:

Of course since , this vector field is conservative so that by the closed loop theorem the integral is zero as well. The point here is that we no longer have to worry about which theorem to apply: “closed loop in 2D” means “apply Green’s theorem”.

* Evaluate , C the boundary of the triangle (0,0), (1,0), and (0,1).

Again it’s a closed curve and 2D vector field, so we use Green: and so that and . Thus:

In this case Green’s theorem simplified the integration problem a lot, but that is not always the case:

* Evaluate , where C bounds the region between and . It is clearly a closed curve so with and we have:

In this case we still had to perform some integration but had we tried to evaluate the work via the original paths, we’d have to compute two integrals: one where the path goes along the x-axis from -2 to 2, and the second where the path goes along the parabola . So, while using Green’s theorem did not save that much work, it at least saved us from considering two integrals.

**HW Exercises**

1. , where C is the square with corners (0,0) and (1,1)
2. , C triangle (0,0), (1,3),(0,3)
3. , C circle of radius 2
4. , C the boundary of the region between and
5. , C the ellipse given by

Evaluate , C the unit circle, first directly, then using Green’s theorem