Panel 1

Last Time:

$\vec{F} = \langle M, N \rangle$ conservative $\Rightarrow$ $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$

$\vec{F} = \langle M, N, P \rangle$ conservative $\Rightarrow$ $\operatorname{curl}(\vec{F}) = \vec{\nabla} \times \vec{F} \equiv 0$

How to find potential functions

Find potential for $\vec{F}(x,y) = \langle y\cos(x) + 2xy,\ \sin(x) + x^2 \rangle$

Let $\vec{F}(x,y,z) = \langle y+z,\ x+z,\ y+x \rangle$. Find potential.

Panel 2

$\vec{F}(x,y,z) = \langle y+z,\ x+z,\ y+x \rangle$.

$f_x = y+z \Rightarrow f = yx + zx + C(y,z)$

$f_y = \cancel{x} + C_y(y,z) \overset{\text{want}}{=} \cancel{x} + z \Rightarrow C = zy + D(z)$

$f = yx + zx + zy + D(z)$

$f_z = \cancel{x} + \cancel{y} + D'(z) \overset{\text{want}}{=} \cancel{x} + \cancel{y} \Rightarrow D(z) = c$

$\underline{f(x,y,z) = xy + xz + zy + c}$

Panel 3

Occasionally it is possible to guess a potential function:

Gravity field: $\vec{F} = \left\langle \frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$

Potential function: $f(x,y,z) =$

$f \quad (x^2+y^2+z^2)^{-1/2}$

$f_x = (x^2+y^2+z^2)^{-3/2}\left(-\frac{1}{\cancel{2}}\right)\cancel{2}x$

$f_y = (\quad)^{-3/2}\left(-\frac{1}{\cancel{2}}\right)\cancel{2}y$

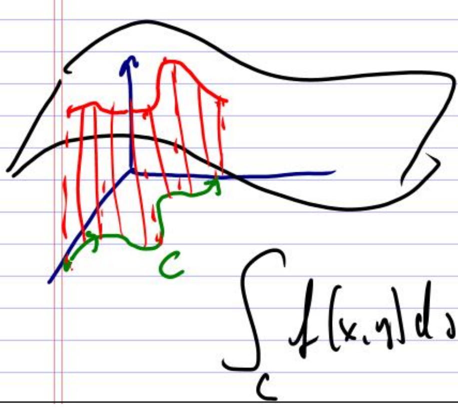
$f_z = (\quad)^{-3/2}\left(-\frac{1}{\cancel{2}}\right)\cancel{2}z$

$f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$ is the potential to Gravity field which is thusly conservative!

Panel 4

Line Integrals

Suppose $r(t) = \langle x(t), y(t) \rangle$ describes a curve C in $\mathbb{R}^2$ and $f(x,y)$ is a function defined on C. Then we define the line integral of f along C as:

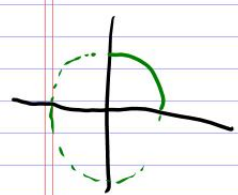


(s is length of the curve $\to ds = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$)

$$\int_C f(x,y)\, ds = \int f(x(t), y(t)) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Panel 5

Ex: Find $\int x y^2 \, ds$, where $\vec{r}$ describes first quarter circle, radius 1.



$r(t) = \langle \cos(t), \sin(t) \rangle, \ t \in [0, \pi/2]$

$$\int_C x y^2 \, ds = \int_0^{\pi/2} (\cos(t))(\sin(t))^2 \sqrt{(-\sin(t))^2 + (\cos(t))^2} \, dt$$

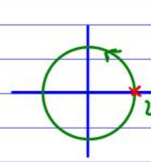
$$= \int_0^{\pi/2} \cos(t) \sin^2(t) \, dt = \frac{1}{3} \sin^3(t) \Big|_0^{\pi/2} = \frac{1}{3}$$

$u = \sin(t)$ $\qquad \int u^2 \, du$

$du = \cos t \, dt$

Panel 6

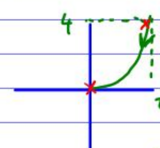
Before we continue, need to find paths: Find $r(t) = \langle x(t), y(t) \rangle$ expressions for these paths:



$r(t) = \langle 2\cos(t), 2\sin(t) \rangle$

$\langle 2\sin(t), 2\cos(t) \rangle$

wrong way

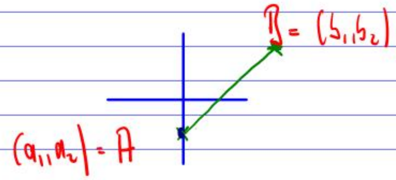


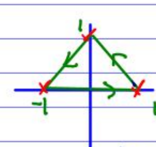
$f(x) = y = x^2$

$\langle t, t^2 \rangle$

Let $x = t$.

$\langle t, f(t) \rangle$





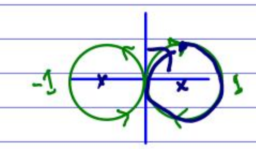
3 curves, each linear

$\ell(t) = (a_1, a_2) + t \langle b_1 - a_1, b_2 - a_2 \rangle$

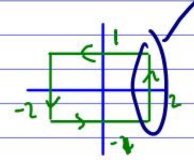
Panel 7

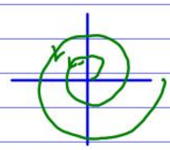
Before we continue, need to find paths: Find $r(t) = \langle x(t), y(t) \rangle$ expressions for these paths:

2 circles: 1^{st} $r(t) = \langle \frac{1}{2} + \frac{1}{2}\sin(t), 0 + \frac{1}{2}\cos(t) \rangle$



$r_1(t) = \langle 2, t \rangle$, $t \in [-1, 1]$

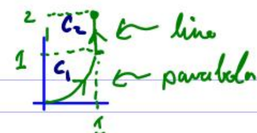




$\langle t\cos(t), t\sin(t) \rangle$

Panel 8

Ex: Evaluate $\int_C 2x\, ds$ where C:



$$\int_{C_1} 2x\, ds = \int_0^1 2t\sqrt{1+4t^2}\, dt = \frac{2}{3}\cdot 2 \cdot \frac{1}{8}\left(1+4t^2\right)^{3/2}\Big|_0^1 = [illegible]$$

$C_1: r(t) = \langle t, t^2 \rangle$, $t \in [0, 1]$

$$\int_{C_2} 2x\, ds = \int_1^2 2\sqrt{1}\, dt = 2t\Big|_1^2 = \underline{4}$$

+

$C_2: r(t) = \langle 1, t \rangle$, $t \in [1, 2]$

Panel 9

We also define two variations of line integrals:

C is curve $(x(t), y(t))$, $t \in [a,b]$.

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{(x')^2+(y')^2}\,dt$$

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

$\cancel{\int_a^b g(x)\,dx}$ old news

Panel 10

Ex: Find $\int_C xy^2\,dx$ and $\int_C xy^2\,dy$ where C is parabola from $(0,0)$ to $(2,4)$

$r(t) = \langle t, t^2 \rangle$, $t \in [0,2]$

$$\int_C xy^2\,dx = \int_0^2 t(t^2)^2 \cdot 1\,dt = \int_0^2 t^5\,dt = \underline{\tfrac{1}{6}2^6}$$

$$\int_C xy^2\,dy = \int_0^2 t(t^2)^2\,2t\,dt = 2\int_0^2 t^6\,dt = \underline{\tfrac{2}{7}2^7}$$

Panel 11

Ex: Evaluate $\int_C xy\,dx + x^2\,dy$ if

$C_1: \langle 3t-1, \underbrace{3t^2-2t}\rangle,\ 1 \le t \le 5/3$

C_2: line segment from $(2,1)$ to $(4,5)$

$$C_1 \quad \int xy\,dx = \int_1^{5/3} \underbrace{(3t-1)(3t^2-2t)\,\overbrace{3\,dt}^{dx}}$$

$$\int x^2\,dy = \int (3t-1)^2(6t-2)\,dt$$

$$C_1: \int xy\,dx + x^2\,dy = \int_1^{5/3} (3t-1)(3t^2-2t)\cdot 3 + (3t-1)^2(6t-2)\,dt$$

Panel 12

Def: C a curve $r(t) = \langle x(t), y(t)\rangle$, $t \in [a,b]$

$\Rightarrow \int_C f(x,y)\,ds = \int f(x(t),y(t))\sqrt{x'^2+y'^2}\,dt$

(line integral of f along curve C)

(*) $\Rightarrow \int_C f(x,y)\,dx = \int f(x(t),y(t))\frac{dx}{dt}\,dt$

(line integral of f along curve C with respect to x)

(**) $\Rightarrow \int_C f(x,y)\,dy = \int f(x(t),y(t))\,y'\,dt$

(line integral of f along curve C with respect to y)

$\Rightarrow \int_C f(x,y)\,dx + g(x,y)\,dy =$ (*) + (**)

Panel 13

Line Integrals of Vector Fields:

Suppose $\vec{F}$ is a vector field on a smooth curve C, defined via $\vec{r}(t)$, $a \le t \le b$. Then the line integral of $\vec{F}$ along C is:

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b F(r(t)) \cdot r'(t)\, dt = \int_a^b M\,dx + N\,dy$$

Note: $\vec{F} = \langle M, N \rangle$, $d\vec{r} = \langle dx, dy \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C M\,dx + N\,dy$$

Panel 14

Ex: Let $\vec{F}(x,y) = \langle x^2, -xy \rangle$ C quarter circle radius 1.

Find $\int_C \vec{F}\, d\vec{r}$ $\quad r(t) = \langle \cos(t), \sin(t) \rangle$, $t \in [0, \frac{\pi}{2}]$

$$\int \vec{F} \cdot d\vec{r} = \int \langle x^2, -xy \rangle \cdot \langle dx, dy \rangle =$$

$$= \int x^2\,dx - xy\,dy =$$

$$= \int_0^{\pi/2} -\cos^2(t)\sin(t) - \cos^2(t)\sin(t)\,dt =$$

$$= -2\int_0^{\pi/2} \sin(t)\cos^2(t)\,dt = -\frac{2}{3} \cdot -\cos^3(t)\Big|_0^{\pi/2} = \frac{2}{3}$$

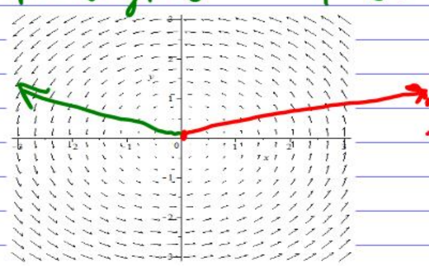
Panel 15

Physics Interpretation of Line Integral

$$\int_C \vec{F}\, d\vec{r} = \int_a^b M\,dx + N\,dy + P\,dz$$

gives the Work done by moving a particle through the (force) field $\vec{F}$ along path C!

$\vec{F} = \langle -y, x \rangle$, C: line from $(0,0)$ to $(5,1)$



$\int \vec{F}\, d\vec{r}$ $+$

$\int \vec{F}\, d\vec{r}$ $-$