Math 2511 – Calc 3 Practice Exam 1

This is a practice exam. The actual exam consists of questions of the type found in this practice exam, but will be shorter. If you have questions do not hesitate to send me email.

1. **Definitions**: Please state in your own words the meaning of the following terms:
   a) Vector
   b) Dot product, Cross product
   c) Angle between two vectors
   d) Unit vector
   e) Projection of **v** onto **w**
   f) Plane, Line, Space curve
   g) Distance between plane and point
   h) Intersection between a line and a point
   i) Smooth curve
   j) Tangent vector to a curve
   k) Unit tangent vector to a curve
   l) Normal vector to a curve
   m) Binormal vector
   n) Curvature
   o) Length of a curve

   look up in book or notes

2. **True/False** questions:
   T a) $u \cdot u = \|u\|^2$
   T b) $<1,3,2>$ and $<-4,-2,5>$ are perpendicular   dot product is zero
   F c) $<1,3,-2>$ and $<2,6,4>$ are parallel
   F d) P(1,2,3) is on the plane 3x – 2y – z = -2 $3\cdot 1-2\cdot 2-3 \neq -2$
   F e) P(1,2,3) is on the line $r(t) =< 1+2t, 2+4t, 1-t >$ $r(0) = (1,2,1) \neq (1,2,3)$
   T f) $v \times w = -w \times v$
   F g) $v \cdot w = \|v\|\,\|w\| \cancel{\sin}\ \cos(\theta)$
   F h) $\frac{d}{dt}\|r(t)\| = \left\|\frac{d}{dt} r(t)\right\|$
   F i) $\frac{d}{dt} p(t) \times r(t) = p'(t) \times r'(t)$
   F j) $r(t) = \left\langle \sqrt{t}+2, 3-\sqrt[3]{t}, \sqrt[4]{t} \right\rangle$ is the equation of a line
   F k) If $\|r(t)\| \equiv 1$ then $r(t) \times r'(t) = 0$
   T l) The planes $x+3y+2z=5$ and $4x+2y-5z=0$ are perpendicular $<1,3,2>\cdot<4,2,-5>=0$
   T m) The distance between $x-y+z=2$ and $x+y+z=1$ is zero   not parallel so

3. **Vectors**: Suppose $u =< 7,-2,3 >$, $v =< -1,4,5 >$, and $w =< -2,1,-3 >$
   a) Are u and v orthogonal, parallel, or neither?

   $<7,-2,3>\cdot<-1,4,5> = -7-8+15 = 0$

   b) Find graphically and algebraically $2u+3v$ and $u-v$

   Hard to do in 3D but easy in 2D ⇒ practice this kind in 2D !!

c) Find the angle between v and w

$$\cos(\theta) = \frac{\langle -1,4,5\rangle \cdot \langle -2,1,-3\rangle}{\sqrt{42}\,\sqrt{14}} = \frac{2+4-15}{\sqrt{42}\sqrt{14}} = \frac{-9}{\sqrt{42}\sqrt{14}}$$

d) Find $u \cdot v$ (dot product), $u \times v$ (cross product), $u \cdot (v \times w)$, and $\|u\|$

$u \cdot v = 0$

$u \cdot (v \times w) = -72$

$u \times v = \langle -22, -38, 26\rangle$

$\|u\| = \sqrt{62}$

e) Find the projection of w onto u and the projection of u onto w

$$\text{proj}_u(w) = \frac{u \cdot w}{\|u\|^2} \cdot u = \frac{-25}{(\sqrt{62})^2} \cdot u, \quad ; \quad \text{proj}_w(u) = \frac{-25}{(\sqrt{14})^2} w$$

4. **Lines and Planes**

a) Find the equation of the plane spanned by $\underset{v}{<1,3,-2>}$ and $\underset{w}{<2,1,2>}$ through the point $P(1,2,3)$

$v \times w = \langle 8, -6, -5\rangle \Rightarrow 8x - 6y - 5z + D = 0$

$D = 19 \Rightarrow \underline{8x - 6y - 5z + 19 = 0}$

b) Find the equation of the plane through $P(1,2,3)$, $Q(1,-1,1)$, and $R(3,2,1)$

$PQ \times PR = \langle 6, -4, 6\rangle \Rightarrow 6x - 4y + 6z + D = 0$

$D = -16$

$\underline{6x - 4y + 6z - 16 = 0}$

c) Find the equation of the plane parallel to $x - y + z = 2$ through $P(0,2,0)$

$x - y + z + D = 0$

$D = 2$

$\Rightarrow \underline{x - y + z + 2 = 0}$

d) Find the equation of a plane parallel to the lines $l_1(t) =< 1 - 2t, 2t, 3 - t >$ and $l_2(t) =< t, 1 - 2t, 2 + 2t >$ through the point P(1, 0, 0)

$\langle -2, 2, -1\rangle \times \langle 1, -2, 2\rangle = \langle 2, 3, 2\rangle$

$\Rightarrow 2x + 3y + 2z + D = 0$

$2 + D = 0 \Rightarrow D = -2$

$\underline{2x + 3y + 2z - 2 = 0}$

e) Find the equation of the line through $P(1,2,3)$ and $Q(1,-1,1)$

$l(t) = (1,2,3) + t\langle 0,-3,-2\rangle$

f) Find the line parallel to the line $l(t) = < 1-2t, 2t, 3-t >$ through P(1,1,1)

$l(t) = (1,1,1) + t(-2,2,-1)$.

5. **Distances**

a) Find the distance between the points P(1,2,3) and Q(1,2,5)

$$d = \sqrt{(1-1)^2+(2-2)^2+(5-3)^2} = 2$$

b) Find the distance between the line $x - y = 2$ and $P(1,2)$

$$\frac{ax_0+by_0+c}{\|\langle a,b\rangle\|} = \frac{|1\cdot 1 - 1\cdot 2 - 2|}{\sqrt{2}} = 3/\sqrt{2}$$

c) Find the distance between the line $l(t) = < 1-2t, 2t, 3-t >$ and the point *P(1, 2, 3)*

$Q_1(1,0,3)$ and $Q_2(-1,2,2)$ are on line.

$$d = \frac{\|\overrightarrow{Q_1Q_2} \times \overrightarrow{Q_1P}\|}{\|\overrightarrow{Q_1Q_2}\|} = \frac{\|\langle 2,0,-4\rangle\|}{\|\langle -2,2,-1\rangle\|} = \frac{\sqrt{20}}{\sqrt{9}} = \frac{\sqrt{20}}{3}$$

d) Find the distance between the plane $x + y + z = 1$ and the point *P(1, 2, 3)*

$$d = \frac{1\cdot 1+1\cdot 2+1\cdot 3-1}{\sqrt{3}} = 5/\sqrt{3}$$

e) Find the distance between the plane $x + 2y - z = 1$ and the line $l(t) = < 4, -1, 2 >$

not a line,
so nothing to do!

f) Find the distance between the planes $x-y+z=2$ and $2x-2y+2z=5$

Planes are parallel, so distance is not zero

P on plane I: $\underline{P(2,0,0)}$

$$\Rightarrow d = \frac{|2\cdot 2-0-0-5|}{\sqrt{2^2+2^2+2^2}} = \underline{\frac{1}{\sqrt{12}}}$$

6. **Intersections**

a) Are the following lines parallel, skew, or intersecting? If they intersect, find the point of intersection: $l_1(t) =< 2,-1,2 > +t < 3,1,1 >$ and $l_2(t) =< -1,0,0 > +t < 3,-1,2 >$

$2+3t = -1+3s$

$-1+t = -s \quad \Rightarrow s = 1-t$

$2+t = 2s$

$2+t = 2(1-t) \Rightarrow 2+t = 2-t \Rightarrow \underline{t=0}$

$\Rightarrow \underline{s=1}$ checks out for all equations

$\Rightarrow$ point of intersection is $\underline{(2,-1,2)}$

b) Do the plane $x-y+z=2$ and the line $l(t) =< 1+t, 2t, 1-5t >$ intersect? If so, where?

$(1+t)-(2t)+(1-5t)=2$

$t-2t-5t = 2-1-1$

$\Rightarrow 6t = 0 \quad \Rightarrow t=0$

point of intersection:

$\underline{P(1,0,1)}$

c) Do the planes $x-y+z=0$ and $2x-z=3$ intersect? If so, find the line of intersection.

?

$\langle 1,-1,1\rangle \times \langle 2,0,1\rangle = \langle -1,1,2\rangle$ is dir of line of intersection.

Goes through: $x=0$: $-y+z=0$, $-z=3$ $\Rightarrow \underline{x=0, y=-3, z=-3}$

$\underline{l(t) = (0,-3,-3)+t\langle -1,1,2\rangle}$ is line of intersection

7. **Vector valued functions**:

a) Find $r'(t)$ if $r(t) =< 6t, -7t^2, t^3 >$

$r'(t) = \langle 6, -14t, 3t^2\rangle$

b) Find $r''(t)$ if $r(t) =< a\cos^3(t), a\sin^3(t), t\sin(t) >$

$r'(t) = \langle -3a\cos^2(t)\sin(t), 3a\sin^2(t)\cos(t), \sin(t)+t\cos(t)\rangle$

$r''(t) = \underline{\text{messy}}$

c) If $r(t) = < 4t, t^2, t^3 >$, find $r'(t)$, $r''(t)$, $\frac{d}{dt}\|r(t)\|$, $\left\|\frac{d}{dt} r(t)\right\|$

$r'(t) = \langle 4, 2t, 3t^2 \rangle$

$r''(t) = \langle 0, 2, 6t \rangle$

$\frac{d}{dt}\|r(t)\| = \frac{d}{dt}\sqrt{16t^2 + t^4 + t^6} = \frac{1}{2}(16t^2+t^4+t^6)^{-1/2} \cdot (32t + 4t^3 + 6t^5)$

$\left\|\frac{d}{dt} r\right\| = \|\langle 4, 2t, 3t^2 \rangle\| = \sqrt{16 + 4t^2 + 9t^4}$

d) If $r(t) = < e^t, 3t^3, \frac{3}{6t} >$ some curve, find $\int_1^2 r(t)dt$ and $\int_1^2 \|r'(t)\|\ dt$

$\int_1^2 r(t)dt = \langle \int_1^2 e^t dt, \int_1^2 3t^3 dt, \int_1^2 \frac{3}{3t} dt \rangle = \langle e^2 - e, \frac{45}{4}, \frac{1}{2}\ln(2) - 0 \rangle =$

$= \langle e^2 - e, \frac{45}{4}, 2\ln(2) \rangle$

e) If $r(t) = < t, \frac{1}{t} >$, find $T(t)$, $N(t)$

$r'(t) = \langle 1, -\frac{1}{t^2} \rangle$, $\|r'\| = \sqrt{1 + 1/t^4} \Rightarrow T(t) = \frac{1}{\sqrt{1+t^{-4}}} \langle 1, -1/t^2 \rangle$

$N =$ (2D trick discussed in class) $= \pm \frac{1}{\sqrt{1+t^{-4}}} \langle 1/t^2, 1 \rangle$

either plus or minus

f) Repeat (e) for $r(t) = < e^t \cos(t), e^t \sin(t) >$ for $t = \frac{\pi}{2}$ (Hint: there is a neat trick to find N(t), which only works in 2 dimensions)

$r'(t) = \langle e^t \cos(t) - e^t \sin(t), e^t \sin(t) + e^t \cos(t) \rangle$

$\|r'(t)\| = \sqrt{2}e^t \Rightarrow T(t) = \frac{1}{\sqrt{2}} \langle \cos(t) - \sin(t), \sin(t) + \cos(t) \rangle$

$\Rightarrow T(\pi/2) = \frac{1}{\sqrt{2}} \langle -1, 1 \rangle$

$\Rightarrow N(t) = \pm \frac{1}{\sqrt{2}} \langle 1, 1 \rangle$ plus or minus

g) Repeat (e) for $r(t) = < 2t, 1, t^3 >$ and also find the binormal at t = 1.

$$r(t) = \langle 2t, 1, t^3 \rangle \Rightarrow r'(t) = \langle 2, 0, 3t^2 \rangle \Rightarrow \|r'(t)\| = \sqrt{4+9t^4}$$

$$\Rightarrow T(t) = (4+9t^4)^{-1/2} \langle 2, 0, 3t^2 \rangle \qquad \Rightarrow T(1) = \frac{1}{\sqrt{13}} \langle 2, 0, 3 \rangle$$

$$\Rightarrow T'(t) = -\frac{1}{2}(4+9t^4)^{-3/2} \cdot (36t^3) \langle 2, 0, 3t^2 \rangle + (4+9t^4)^{-1/2} \langle 0, 0, 6t \rangle$$

$$= (4+9t^4)^{-1/2} \left[\left\langle \frac{-36t^3}{4+9t^4}, 0, \frac{-54t^5}{4+9t^4} \right\rangle + \langle 0, 0, 6t \rangle \right] = (4+9t^4)^{-1/2} \left\langle \frac{-36t^3}{4+9t^4}, 0, \frac{\cancel{-54t^5} + 24t \cancel{+ 54t^5}}{4+9t^4} \right\rangle$$

$$= (4+9t^4)^{-3/2} \langle -36t^3, 0, 24t \rangle \Rightarrow \|T'\| = (4+9t^4)^{-3/2} \cdot 12t\sqrt{4+9t^4} = \frac{12t}{4+9t^4}$$

$$\Rightarrow N(t) = (4+9t^4)^{-3/2} \langle -36t^3, 0, 24t \rangle \frac{(4+9t^4)}{12t} = (4+9t^4)^{-1/2} \langle -3t^2, 0, 2 \rangle \Rightarrow N(1) = \frac{1}{\sqrt{13}} \langle -3, 0, 2 \rangle$$

$$\Rightarrow B = T \times N = \langle 0, -1, 0 \rangle$$

h) If $r(t) = \langle 3-3t, 4t \rangle$, find the arc length of the curve between 0 and 1

$$L = \int_0^1 \|r'\| dt = \int_0^1 \sqrt{9+16} = 5 \cdot (1-0) = 5$$

i) If $r(t) = \langle 4t, 3\cos(t), 3\sin(t) \rangle$, find the arc length of the curve between 0 and $\frac{\pi}{2}$

$$L = \int_0^{\pi/2} \|r'\| dt = \int_0^{\pi/2} \sqrt{16+9}\, dt = \frac{5\pi}{2}$$

j) If $r(t) = \langle 2t^2, 3t-1, \cos(t) \rangle$, find the arc length of the curve between 0 and π (Hint: if the integration becomes tricky, try Wolfram Alpha)

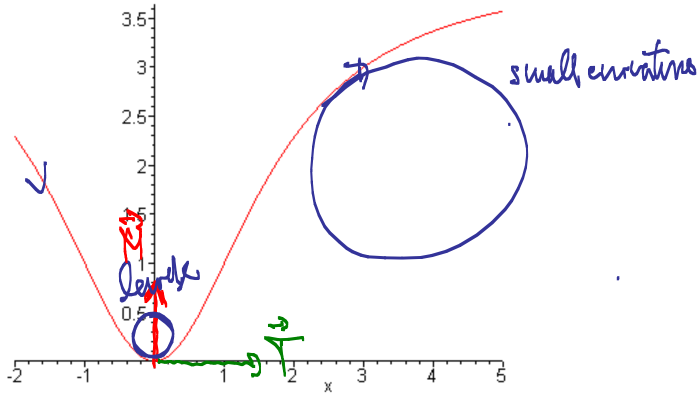
$$L = \int_0^{\pi} \|r'\| dt = \int_0^{\pi} \sqrt{16t^2 + 9 + \sin^2(t)}\, dt = 22.8$$

k) Find the curvature of $r(t) = \langle t, 3t^2, \frac{t^2}{2} \rangle$

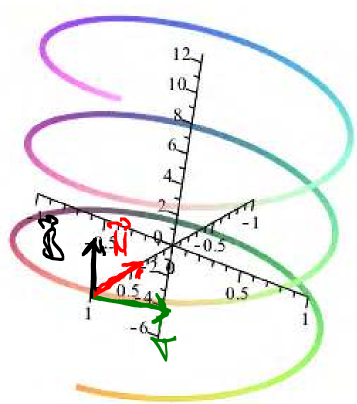
$$r'(t) = \langle 1, 6t, t \rangle \qquad r''(t) = \langle 0, 6, 1 \rangle$$

$$\kappa = \frac{\|r' \times r''\|}{\|r'\|^3} = \frac{\|\langle 0, -1, 6 \rangle\|}{(\sqrt{1+37t^2})^3} = \frac{\sqrt{37}}{(1+37t^2)^{3/2}}$$

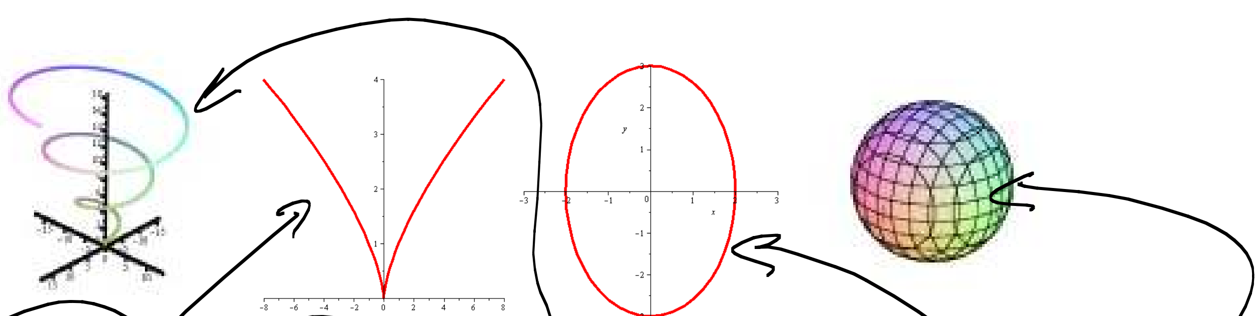
8. **Picture:** Sketch the circle that fits the graph below the best at the points (0,0) and *(3,3)*. At which of the two points is the curvature smaller? Also sketch the unit tangent and unit normal for *(3,3)*.



9. **Picture:** Sketch the unit tangent, normal, and binormal to the curve <cos(t), sin(t), t> as best as possible, when time *t =0*. A sketch suffices, you don't have to compute the actual vectors:



10. **Picture**: Match the following functions to their corresponding plots.



$r(t) = < t^3, t^2 >$ $x^2 + y^2 + z^2 = 1$ $r(t) = < t\cos(t), t\sin(t), t >$ $r(t) = < 2\sin(t), 3\cos(t) >$

12. **Prove** the following facts:

a) Show that $u \times v = -(v \times u)$ $u = \langle u_1, u_2, u_3 \rangle$, $v = \langle v_1, v_2, v_3 \rangle$

$\Rightarrow u \times v = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle$

$v \times u = \langle v_2 u_3 - u_2 v_3, u_1 v_3 - v_1 u_3, u_2 v_1 - u_1 v_2 \rangle$

$\Rightarrow u \times v = -v \times u$.

b) Show that $u \cdot (v \times u) = 0$

Use as above $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$ and work out $u \cdot (v \times u)$ carefully.

c) Show that if $y = f(x)$ is a function that is twice continuously differentiable, then the curvature of f at a point x is $K = \dfrac{|f''(x)|}{\left(1 + [f'(x)]^2\right)^{3/2}}$

$y = f(x) \Rightarrow$

$r(t) = \langle t, f(t), 0 \rangle$

$\Rightarrow r'(t) = \langle 1, f'(t), 0 \rangle$

$r''(t) = \langle 0, f''(t), 0 \rangle$

$\Rightarrow \dfrac{\| r' \times r'' \|}{\| r' \|^3} = \dfrac{|f''(t)|}{\left(1 + [f'(t)]^2\right)^{3/2}}$

d) Prove that the curvature of a line in space is zero

$l(t) = \langle a, b, c \rangle + t \langle v_1, v_2, v_3 \rangle =$

$l'(t) = \langle v_1, v_2, v_3 \rangle$

$l''(t) = \langle 0, 0, 0 \rangle$

$\Rightarrow \kappa = \dfrac{\| l' \times l'' \|}{\| l' \|^3} = 0$

13. Here are a few **miscellaneous** questions:

a) In the definition of unit tangent to a curve r(t) we specify that the curve must be smooth. Why?

$T = r' / \|r'\|$ If curve was not smooth, $\|r'\| = 0 \Rightarrow$ divide by zero error!

b) If two lines in three dimensional space are not parallel, do they have to intersect?

No.

d) The curvature of a parabola is largest at its vertex, as we mentioned in class. Where, do you think, is the curvature of a simple 3rd degree polynomial $y = x^3$ the largest?