Panel 1

Last Time: C a curve $\underline{r}(t) = \langle x(t), y(t) \rangle$, $t \in [a,b]$

$$\Rightarrow \int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{(x')^2 + (y')^2}\, dt$$

(line integral of f along curve C)

$$\Rightarrow \int_C f(x,y)\, dx = \int_a^b f(x(t), y(t)) \frac{dx}{dt} \cdot dt$$

(line integral of f along curve C with respect to x)

$$\Rightarrow \int_C f(x,y)\, dy = \int_a^b f(x(t), y(t)) \frac{dy}{dt}\, dt$$

(line integral of f along curve C with respect to y)

$$\Rightarrow \int_C f(x,y)\, dx + g(x,y)\, dy = \int_C f(x,y)\, dx + \int_C g(x,y)\, dy$$

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Panel 2

Line Integrals of Vector Fields:

Suppose $\vec{F}$ is a vector field on a smooth curve C, defined via $\vec{r}(t)$, $a \le t \le b$. Then the line integral of $\vec{F}$ along C is:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F}(x(t), y(t)) \cdot \vec{r}'(t)\, dt$$

Note: $$\int_C \vec{F}\, d\vec{r} = \int \langle M, N \rangle \cdot \langle dx, dy \rangle = \int_C M\,dx + N\,dy = \int_C M\,dx + \int_C N\,dy$$

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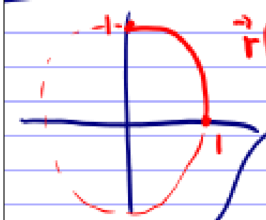
Panel 3

Ex: Let $\vec{F}(x,y) = \langle \overset{M}{x^2}, \overset{N}{-xy} \rangle$, C quarter circle radius 1.

Find $\int_C \vec{F}\, d\vec{r}$

$$\int_C \vec{F}\, d\vec{r} = \int_C M\,dx + N\,dy = \int_C x^2\,dx - xy\,dy =$$



$$\vec{r}(t) = \langle \overset{x}{\cos(t)}, \overset{y}{\sin(t)} \rangle,\ t \in [0, \tfrac{\pi}{2}]$$

$$= \int_0^{\pi/2} \cos^2(t)\overbrace{(-\sin(t))\,dt}^{dx} - \int_0^{\pi/2} \cos(t)^2 \sin(t)\,dt =$$

$$= -2\int_0^{\pi/2} \cos^2(t)\sin(t)\,dt \overset{u=\cos(t)}{=} +2\int_1^0 u^2\,du = \frac{2}{3}u^3\Big|_1^0 = -\frac{2}{3}$$

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Panel 4

Physics Interpretation of Line Integral

$$\int_C \vec{F}\, d\vec{r} = \int_a^b M\,dx + N\,dy + P\,dz$$

$$= \int_a^b M\frac{dx}{dt}\,dt + N\frac{dy}{dt}\,dt + P\frac{dz}{dt}\,dt$$

is work done to move particle along curve C through force field $\vec{F}$

$\vec{F}$ is force field, applied to a path of distance d x

$F \cdot d = $ Work

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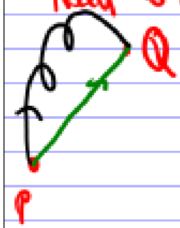
Panel 5

Ex: Find work required to move particle from $(2,0,0)$ to $(3,4,5)$ $\underbrace{\text{}}_{}$ straight line through force field $\vec{F} = \langle y, z, x \rangle$

Need to know: what path from $(2,0,0)$ to $(3,4,5)$?



$$W = \int_C \vec{F}\, d\vec{r} = \int_C y\,dx + z\,dy + x\,dz =$$

$$= \int_0^1 4t \cdot 1\,dt + 5t\, 4\,dt + (2+t)\,5\,dt$$

$C: r(t) = P + t(P-Q) = (2,0,0) + t(1,4,5) = \langle 2+t, 4t, 5t \rangle,\ t \in (0,1)$

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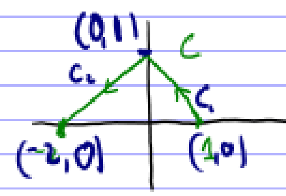
Panel 6

Evaluate $\int_C y^2\,dx - 2x\,dy$ and interprete it as work.



If $\vec{F} = \langle y^2, -2x \rangle$, then I want to compute

$$\int_C \vec{F}\, d\vec{r} = \int_C y^2\,dx - 2x\,dy =$$

work

$$= \int_{C_1} y^2\,dx - 2x\,dy + \int_{C_2} y^2\,dx - 2x\,dy =$$

$$= \int_0^1 t^2(-1)\,dt - 2(1-t)(1)\,dt + \int_0^1 (1-t)^2(-2)\,dt - 2(-2t)(-1)\,dt$$

$C_1: r_1(t) = (1,0) + t(-1,1)$ $= (1-t, t)$

$C_2: r_2(t) = (0,1) + t(-2,-1)$ $(-2t, 1-t)$

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Panel 7

Fundamental Theorem of Line Integrals:

Suppose $\vec{F}$ is a conservative vector field. Then

$$\int_C \vec{F}\, d\vec{r} = f(r(b)) - f(r(a)) = \text{potential at end-pot. at start}$$

where f is the potential of F, i.e. $\nabla f = \vec{F}$

and $r: [a,b] \to \mathbb{R}^2$, $r(t) = \langle x(t), y(t) \rangle$, $t = a$ to b

Note: Looks easy, but you need to know the potential function!!!

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Panel 8

Ex: Find work done by gravitational field $\vec{F}(\vec{r}) = -\frac{mMG}{\|r\|^3}\vec{r}$ moving particle from $(3,4,12)$ to $(2,2,0)$.

Note: no path is given! Work can not depend on path! Thus, $\vec{F}$ better be conservative.

Potential

$\vec{F}(x,y,z) = -\frac{mMG}{(x^2+y^2+z^2)^{3/2}} \cdot (x,y,z)$ Try guess: $f = mMG(x^2+y^2+z^2)^{-1/2}$

check: $\nabla\left(mMG(x^2+y^2+z^2)^{-1/2}\right) = -mMG\left\langle \frac{2x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(\quad)^{3/2}}, \frac{z}{(\quad)^{3/2}}\right\rangle$

Thus: $W = \int_C -\frac{mMG}{\|\vec{r}\|^3}\vec{r} \cdot d\vec{r} = f(2,2,0) - f(3,4,12) = mMG\left(8^{-1/2} - 169^{-1/2}\right)$

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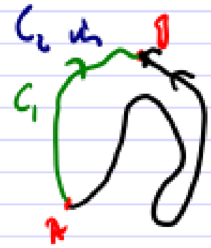
Panel 9

Thm: If $\vec{F} = \nabla f$, i.e. $\vec{F}$ is conservative and C is a smooth curve from A to B then

$$\int_C \vec{F}\, d\vec{r} = f(B) - f(A)$$

Corollary 1: If C_1 curve from A to B and C_2 is another curve from A to B, then



If $\vec{F}$ is conserv.: $$\int_{C_1} \vec{F}\, d\vec{r} = \int_{C_2} \vec{F}\, d\vec{r}$$

Corollary 2: If $\vec{F}$ is conservative and C a closed curve, then $$\oint_C \vec{F}\, d\vec{r} = 0$$

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Panel 10

Integral Soup

$\int_a^b f(x)\,dx$ Calc 1

$\iint_R f(x,y)\,dA$ Fubini

$\iiint_Q f(x,y,z)\,dV$ Fubini

$\int_a^b ds = \int_a^b \sqrt{(x')^2+(y')^2}\,dt$

length of curve

$\int_C f(x,y)\,ds = \int f(x(t),y(t))\sqrt{x'^2+y'^2}\,dt$

$\int_C f(x,y)\,dx = \int f(x(t),y(t))\,x'\,dt$

$\int_C f(x,y)\,dy = \int \quad y'\,dt$

$\int_C \vec{F}\, d\vec{r} = \int M\,dx + \int N\,dy$

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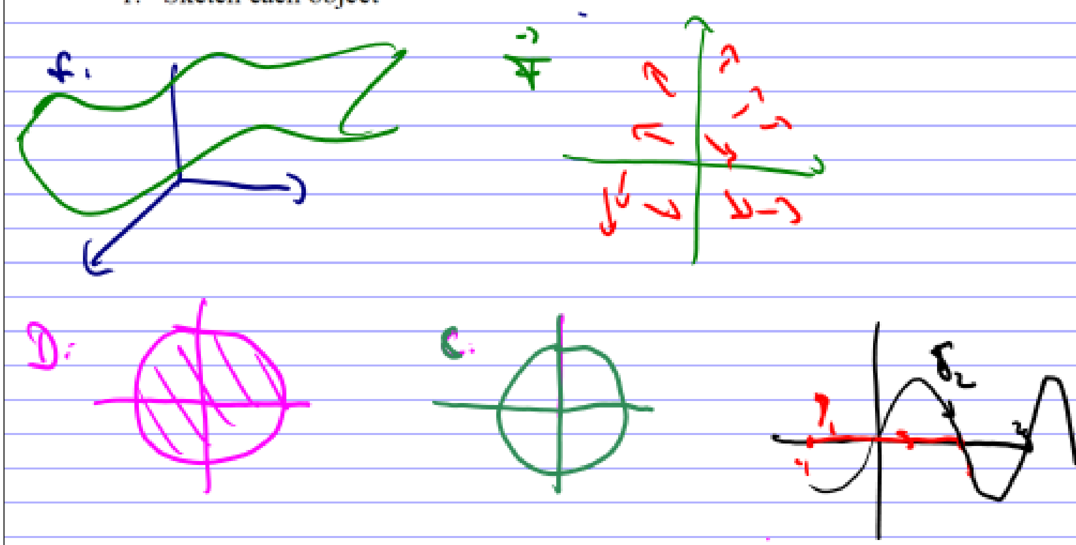
Panel 11

f=

Let $f(x,y)=x^2-xy+y^2$, $F(x,y)=<2x-y,2y-x>$, $D=\{(x,y):x^2+y^2\le 1\}$,
$C=\{(x,y):x^2+y^2=1, y\ge 0\}$, $\gamma_1(t)=<t,0>$, $t\in[-1,1]$, and $\gamma_2(t)=<t,\sin(\pi t)>$, $t\in[-1,1]$.

x, $y=\sin(\pi x)$

1. Sketch each object



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Panel 12

Let $f(x,y)=x^2-xy+y^2$, $\vec F(x,y)=<2x-y,2y-x>$, $D=\{(x,y):x^2+y^2\le 1\}$,
$C=\{(x,y):x^2+y^2=1, y\ge 0\}$, $\gamma_1(t)=<t,0>$, $t\in[-1,1]$, and $\gamma_2(t)=<t,\sin(\pi t)>$, $t\in[-1,1]$.

a) $\iint_D \cancel{f(x,y)}\,dA$

b) $\int_D \cancel{f(x,y)}\,ds$

c) $\int_C f(x,y)\,ds$

d) $\int_{\gamma_1} f(x,y)\,dx$

e) $\int_{\gamma_2} f(x,y)\,dy$

f) $\int_{\gamma_1} \cancel{f(x,y)}\,d\vec r$

g) $\int_{\gamma_1} \vec F\cancel{(x,y)}\,dx$

h) $\int_D \vec F\cancel{(x,y)}\,d\vec r$

i) $\int_C F(x,y)\,d\vec r$

j) $\int_{\gamma_1} F(x,y)\,dr$

k) $\int_{\gamma_2} F(x,y)\,dr$

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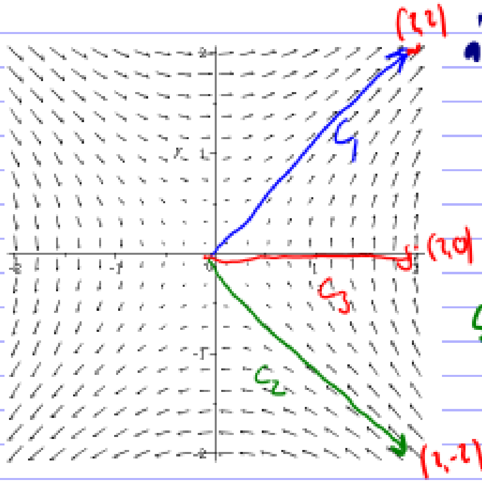
Panel 13

Ex: Are the following integrals positive or negative?



a) $\int_{C_1} \vec{F}\, d\vec{r} = xy\Big|_{(0,0)}^{(2,2)} = 4-0 = \underline{4} > 0$

$\int y\,dx + x\,dy = \int_0^1 2t \cdot 2\,dt + 2t \cdot 2\,dt$

$r_1(t) = (0,0) + (2,2)t$

$= \int_0^1 8t\,dt = \underline{4} > 0$

b) $\int_{C_2} \vec{F}\, d\vec{r} = (xy)\Big|_{(0,0)}^{(2,-2)} = \underline{-4 < 0}$

$\vec{F} = \langle y, x \rangle$

c) $\int_{C_3} \vec{F}\, d\vec{r} = (xy)\Big|_{(0,0)}^{(2,0)} = 0$

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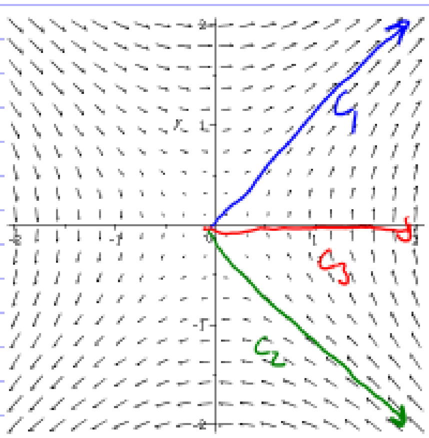
Panel 14

Ex: Why are the following integrals positive or negative?



a) $\int_{C_1} \vec{F}\, d\vec{r}$ is positive, because I go with the flow, don't expend work but get work → positive

b) $\int_{C_2} \vec{F}\, d\vec{r}$ hard work against flow → negative

c) $\int_{C_3} \vec{F}\, d\vec{r}$ is zero because dir. of movement is perp. to force field

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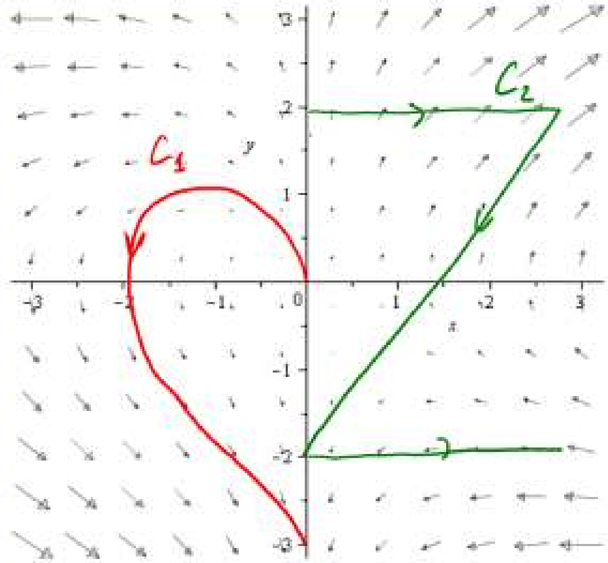
Panel 15

Ex: Are the following integrals positive or negative?



$\int_{C_1} \vec{F}\, d\vec{r}$

pos or neg or zero

$\int_{C_2} \vec{F}\, d\vec{r}$

pos or neg or zero

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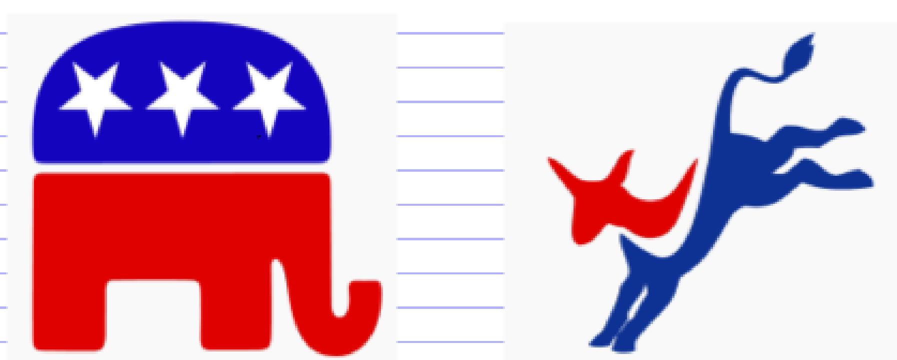
Panel 16

Which of the following vector fields look conservative?



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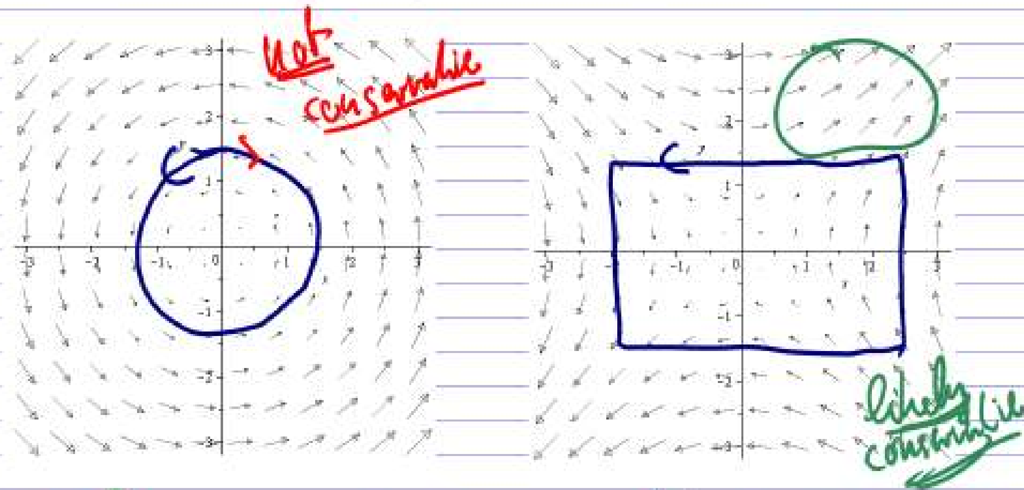
Panel 17

Which of the following vector fields looks conservative?



If $\vec{F}$ is conservative, then $\oint_C \vec{F}\, d\vec{r} = 0$, for every closed curve

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Panel 18

Summary

$\int_C \vec{F}\, d\vec{r}$ is important $\Rightarrow$ gives work

$\int_C \vec{F}\, d\vec{r}$ <
- long way by parametrizing curve
- potential function (if $\vec{F}$ is conservative)

$\oint_C \vec{F}\, d\vec{r}$ <
- long way
- conservative: zero
- other shortcut: next time!

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Panel 19

Ex: Let $F(x,y) = \langle \frac{y^2}{1+x^2}, 2y \arctan(x) \rangle$ and

$r(t) = \langle t^2, 2t \rangle$, $t \in [0,1]$. Find $\int_r \vec{F}\, d\vec{r}$

$$x = t^2, \quad y = 2t$$

① $\int \vec{F}\, d\vec{r} = \int_C \frac{y^2}{1+x^2}\, dx + 2y \arctan(x)\, dy =$

$$= \int_0^1 \frac{(2t)^2}{1+t^4}\, 2t\, dt + 2(2t) \arctan(t^2)\, 2\, dt$$

② $$= y^2 \arctan(x) \Big|_{(0,0)}^{(1,0)} = 0$$

potential $f(x,y) = y^2 \arctan(x)$ ✓

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