Panel 1

Last Time: How to find Relative Extremas

① $\nabla f$ (Gradient)

② Solve $\nabla f = 0$ (can be tricky)

③ $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$, $D = f_{xx} f_{yy} - (f_{xy})^2$

④ $D > 0$, $f_{xx} > 0$ → min

$D > 0$, $f_{xx} < 0$ → max

$D < 0$ → saddle point

$D = 0$ no clue

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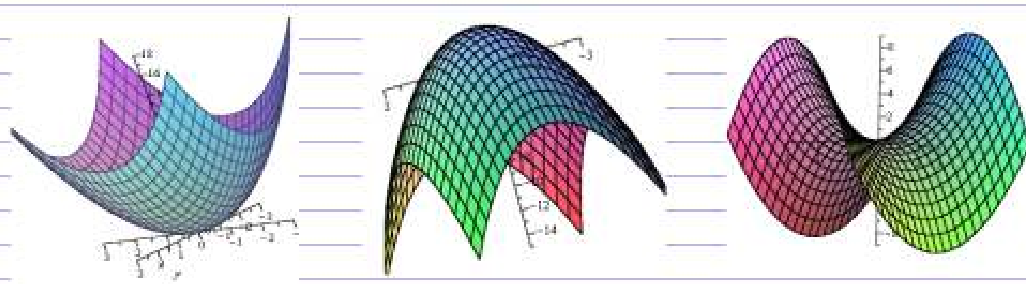
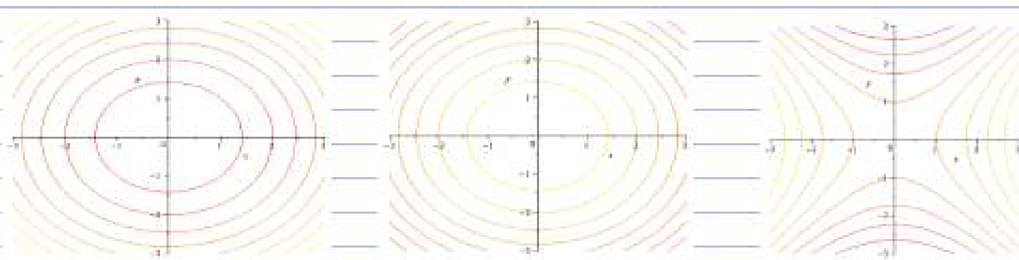
Panel 2



$f(x,y) = x^2 + y^2$ $f(x,y) = 1 - x^2 - y^2$ $f(x,y) = x^2 - y^2$



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Panel 3

$f(x,y) = 9 - 2x + 4y - x^2 - 4y^2$

1 $\nabla f = (f_x, f_y)$ 2

$f_x = -2 - 2x = 0 \quad x = -1$

$f_y = 4 - 8y = 0 \quad y = 1/2$

3 $H = \begin{pmatrix} -2 & 0 \\ 0 & -8 \end{pmatrix}$ 4 $D = 16 > 0, \; f_{xx} < 0 \Rightarrow$ is a max at $(-1, \frac{1}{2})$

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Panel 4

Find and classify critical points for $f(x,y) = 3x - x^3 - 2y^2$

$\nabla f.$ $f_x = 3 - 3x^2 = 0 \Rightarrow x = 1$ or $x = -1$ 2

1 $f_y = -4y = 0 \Rightarrow y = 0$

$\Rightarrow$ Critical points $(1,0), (-1,0)$

3 $H = \begin{pmatrix} -6x & 0 \\ 0 & -4 \end{pmatrix}$ $D = 24x$

at $(1,0)$: $D > 0, \; f_{xx} < 0 \Rightarrow$ max

at $(-1,0)$: $D < 0 \Rightarrow$ saddle point.

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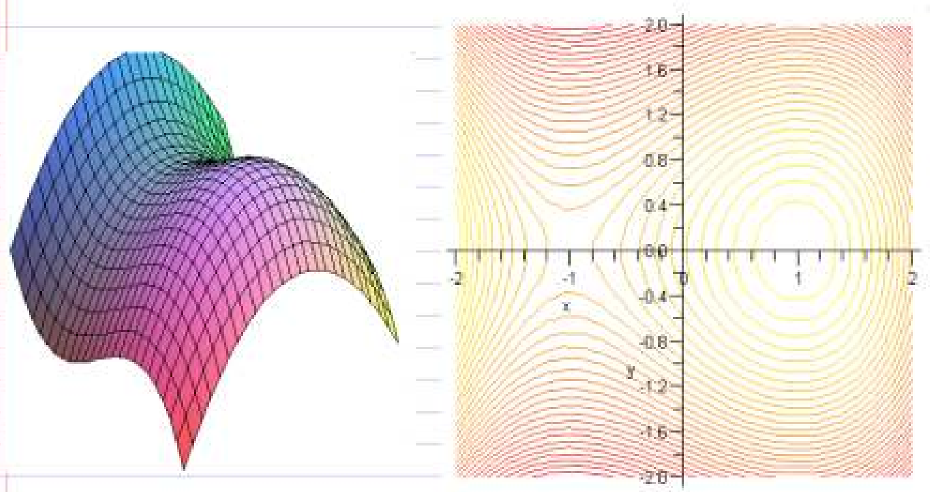
Panel 5

$f(x,y) = 3x - x^3 - 2y^2$

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Panel 6

Ex: Find and classify the critical points for $f(x,y) = x^3y + 12x^2 - 8y$

① $f_x = 3x^2y + 24x = 0$ ② $\longrightarrow 12y + 48 = 0$

$f_y = x^3 - 8 = 0 \Rightarrow x = 2, \; y = -4$

$3x^2y + 24x = x(3xy + 24) = 0 \Rightarrow \underline{x=0}$ does not check 2nd $\underline{eqn}$

$(2,-4)$ is only critical point

$$H = \begin{pmatrix} 6xy + 24 & 3x^2 \\ 3x^2 & 0 \end{pmatrix} \Rightarrow D = 0 - 9x^4 = -9x^4$$

$D$ at $(2,-4)$ is $< 0 \Rightarrow (2,-4)$ is a saddle point!

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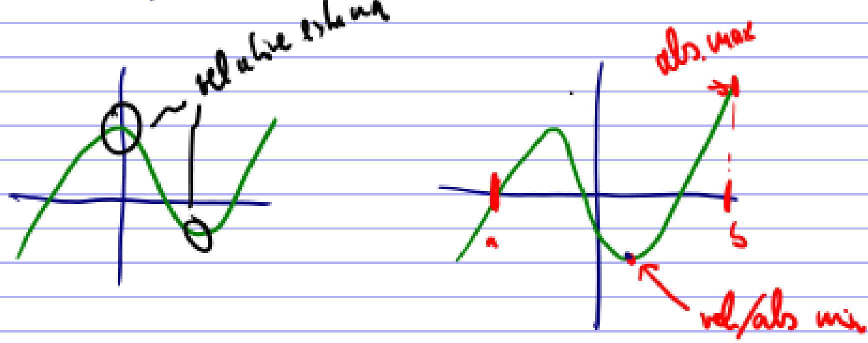
Panel 7

## Absolute Max / Min:

Differences between absolute and relative extrema




**Thm:** If f is cont. on closed bounded set in $\mathbb{R}^2$, e.g. $[a,b]\times[c,d]$. Then f must have abs. max + min, either at a critical point or on the boundary

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Panel 8

## To find abs. max/min

If f is continuous on closed bounded set $D \subset \mathbb{R}^2$:

(1) $\nabla f = 0$ (critical)

(2) Find critical points on boundary ← Work!

(3) Add endpoints (if any)

(4) Eval f at all of these points

→ max + min

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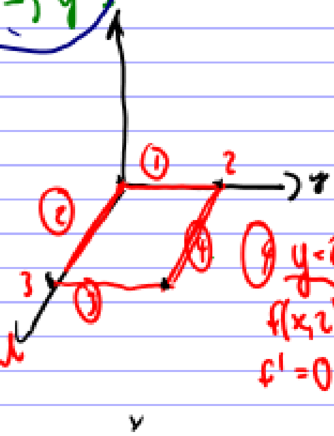
Panel 9

Ex: Find abs. extrema for $f(x,y) = x^2 - 2xy + 2y$ on $[0,3]\times[0,2]$, i.e. $0 \le x \le 3$ and $0 \le y \le 2$

$f_x = 2x - 2y = 0$

$f_y = -2x + 2 = 0$

$\Rightarrow x = 1,\ y = 1$




Consider boundary

(1) $x = 0,\ y \in [0,2]$

$\Rightarrow f(0,y) = 2y$ no critical

(2) $y = 0,\ x \in [0,3]$

$f(x,0) = x^2$, $f'(x) = 2x = 0$, $x = 0, y = 0$

(3) $x = 3,\ y \in [0,2]$

$f(3,y) = 9 - 6y + 2y$

$\Rightarrow$ critical pts

(4) $y = 2,\ x \in [0,3]$:

$f(x,2) = x^2 - 4x + 4$

$f' = 0 \Rightarrow x = 2,\ y = 2$

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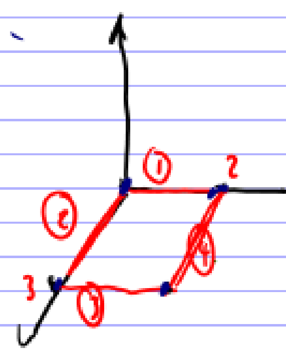
Panel 10

$f(x,y) = x^2 - 2xy + 2y$

| $(x,y)$ | $f$ | |
|---|---|---|
| $\to (0,0)$ | $0$ | $\leftarrow$ min |
| $(3,0)$ | $9$ | $\leftarrow$ max |
| $(3,2)$ | $9 - 12 + 4 = 1$ | |
| $(0,2)$ | $4$ | |
| $(2,2)$ | $4 - 8 + 4 = 0$ | $\leftarrow$ min |
| $(1,1)$ | $1 - 2 + 2 = 1$ | |

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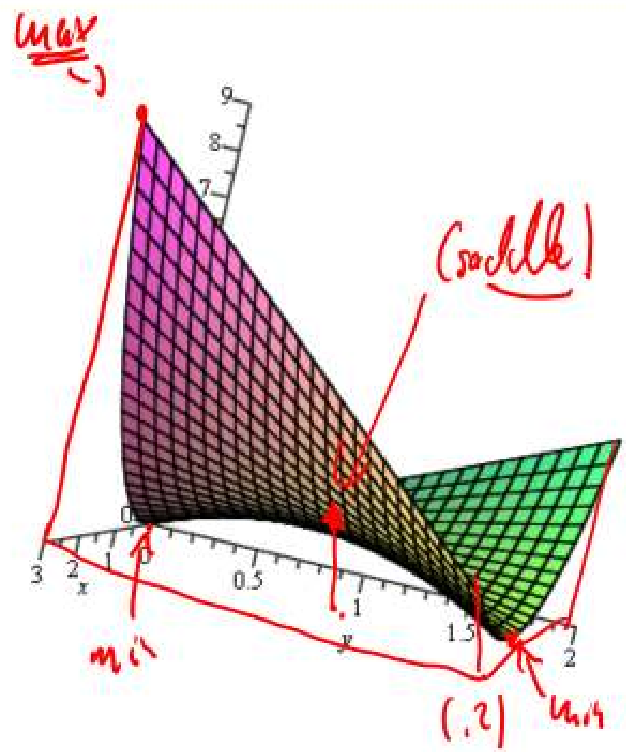
Panel 11




| $(x,y)$ | $z=f(x,y)$ |
|---|---|
| $(1,1)$ | 1 |
| $(2,2)$ | 0 |
| $(0,0)$ | 0 |
| $(0,2)$ | 4 |
| $(3,0)$ | 9 |
| $(3,2)$ | 2 |

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Panel 12

$f(x,y) = x^2 + 2y^2 + 4xy$ for $(x,y) \in [0,1]\times[0,1]$. → abs. extrema?

$f_x = 2x + 4y = 0$

$f_y = 4y + 4x = 0$

$-2x = 0$, $x=0$, $y=0$ critical point

① $x=0$: $f(0,y) = 2y^2 \Rightarrow y=0, x=0$ ✓

② $x=1$: $f(1,y) = 1 + 2y^2 + 4y$, $f' = 4y+4 = 0$, $\cancel{y=-1}$

③ $y=0$: $f(x,0) = x^2 \Rightarrow x=0, y=0$ ✓

④ $y=1$: $f(x,1) = x^2 + 2 + 4x$, $f' = 2x+4 = 0$, $\cancel{x=-2}$

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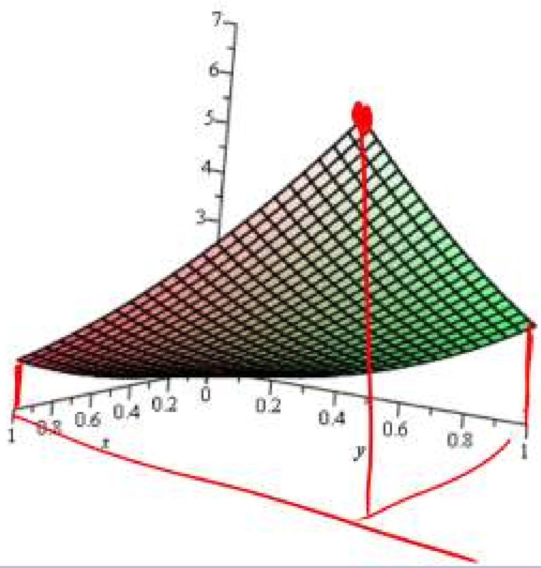
Panel 13

$f(x,y) = x^2 + 2y^2 + 4xy$




| $(x,y)$ | $f(x,y) = z$ |
|---|---|
| $(0,0)$ | 0 |
| $(0,1)$ | 2 |
| $(1,0)$ | 1 |
| $(1,1)$ | 7 |

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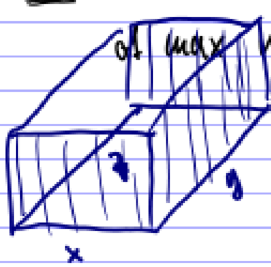
Panel 14

Ex: Make a box w/o lid out of 12 cm² cardboard of max volume.




$$\max V = xyz$$

$$2xz + 2yz + xy = 12$$

$$z(2x+2y) = 12 - xy$$

$$z = \frac{12-xy}{2x+2y}$$

$$V = \frac{xy(12-xy)}{2x+2y}$$

solve $V_x = 0$, $V_y = 0$

find H, D, decide

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Panel 15

Problem: Max $V = xyz$ subject to $2xz + 2yz + xy = 12$

$$\Rightarrow V(x,y) = \frac{xy(12-xy)}{2x+2y}$$

$V(x,y) := \dfrac{xy\,(12 - xy)}{2x + 2y}$

$$(x, y) \to \frac{xy\,(12 - xy)}{2x + 2y} \tag{1}$$

$Vx := diff(V(x, y), x)$

$$\frac{y\,(12 - xy)}{2x + 2y} - \frac{xy^2}{2x + 2y} - \frac{2xy\,(12 - xy)}{(2x + 2y)^2} \tag{2}$$

$Vy := diff(V(x, y), y);$

$$\frac{x\,(12 - xy)}{2x + 2y} - \frac{x^2 y}{2x + 2y} - \frac{2xy\,(12 - xy)}{(2x + 2y)^2} \tag{3}$$

$solve(\{Vx = 0, Vy = 0\}, \{x, y\})$

$$\{x = 2, y = 2\}, \{x = -2, y = -2\} \tag{4}$$

$$z = \frac{12 - xy}{2x + 2y} = \underline{1}$$

(2:01)

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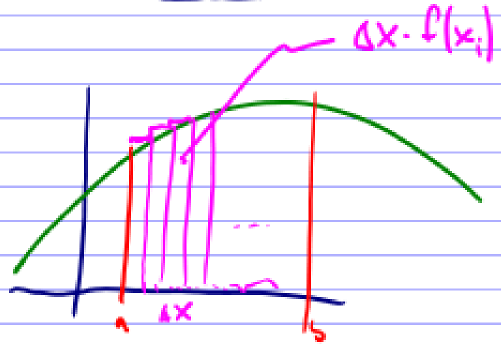
Panel 16

Integration: $\mathbb{R}$




$$A = \int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

= limit of Riemann Sums

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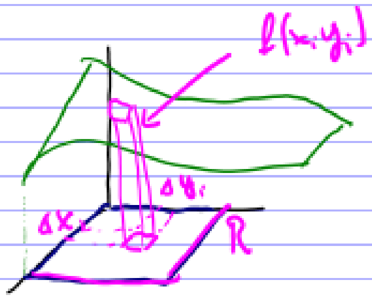
Panel 17

## Integration in $\mathbb{R}^2$




$$\sum_{j=1}^{n}\sum_{i=1}^{n} f(x_i,y_j)\,\Delta x_i\,\Delta y_j$$

$$V = \iint_R f(x,y)\,dA = \lim_{i,j} \sum_{i=1}^{n}\sum_{j=1}^{n} f(x_i,y_j)\,\Delta x_i\,\Delta y_j$$

is the volume under $f(x,y)$ over $R$, as long as $f(x,y) \geq 0$

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Panel 18

## Fubini's Theorem (How to integrate in $\mathbb{R}^2$)

If $f(x,y)$ is continuous on $R = [a,b]\times[c,d]$

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

x y

$(4-16)-(2-2) = -12$

Ex: $\iint_R (x-3y^2)\,dA$, $R = [0,2]\times[1,2]$ $\quad = (y-2y^3)\Big|_1^2 =$

$$= \int_1^2\left(\int_0^2 (x-3y^2)\,dx\right)dy = \int_1^2 \tfrac{1}{2}x^2 - 3y^2x\Big|_{0=x}^{2=x} dy = \int_1^2 2-6y^2-0\,dy =$$

$$= \int_0^2\int_1^2 (x-3y^2)\,dy\,dx = \int_0^2 xy - y^3\Big|_{1=y}^{2=y} dx = \int_0^2 2x-8-(x-1)\,dx =$$

$$= \int_0^2 x-7\,dx = \tfrac{1}{2}x^2 - 7x\Big|_0^2 = -12$$

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