Panel 1

Last Time

Lines: $\ell(t) = \underbrace{\langle x_0, y_0, z_0 \rangle}_{P_0} + t\underbrace{\langle v_1, v_2, v_3 \rangle}_{\vec{v}} = \langle \underbrace{x_0 + v_1 t}_{x}, \underbrace{y_0 + v_2 t}_{y}, \underbrace{z_0 + v_3 t}_{z} \rangle$

Planes: $ax + by + cz + d = 0$

$\vec{n} = \langle a, b, c \rangle$ is <u>normal</u>

Intersections of:

- line + plane: subst line into plane
- two lines: set equal (with different parameters) see if there is solution
- two planes: intersect in a line.

Recall the projection of $\vec{a}$ onto $\vec{b}$: $\text{proj}_{\vec{b}}(\vec{a}) = \dfrac{a \cdot b}{\|b\|^2} \vec{b}$

1

Panel 2

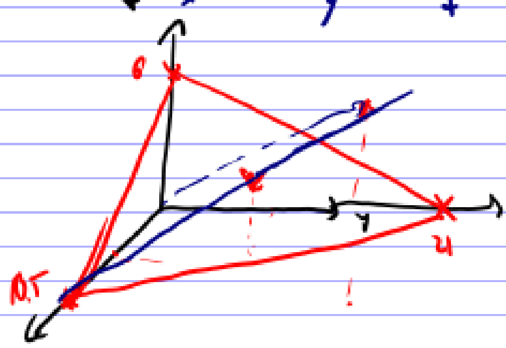
$4x + 2y + 7z = 42$

$\ell(t) = (5, 8, 2) + t\langle 10, 16, 4 \rangle$

$= \langle 5 + 10t, 8 + 16t, 2 + 4t \rangle$

Do they intersect? Not if $\vec{v}$ of line is parallel to plane, i.e.

$\langle 10, 16, 4 \rangle \cdot \langle 4, 2, 7 \rangle \neq 0$



$\to 4(5 + 10t) + 2(8 + 16t) + 7(2 + 4t) = 42$

$20 + 40t + 16 + 32t + 14 + 28t = 42$

$100t = -12$

$t = -0.12$

$\ell(-0.12) = \langle 5, 8, 2 \rangle - 0.12\langle 10, 16, 4 \rangle$

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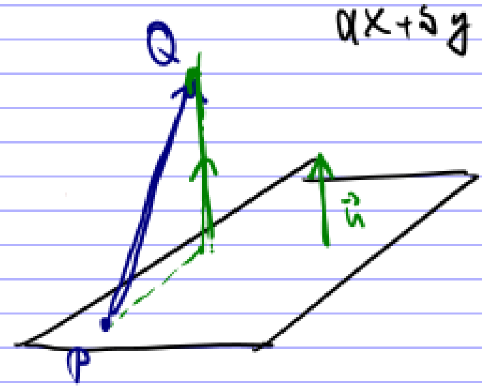
Panel 3

Distance between point $Q(x_0, y_0, z_0)$ and plane

$$ax + by + cz + d = 0$$



1. Find any P on the plane
2. Find $\vec{PQ}$
3. $\| \text{proj}_{\vec{n}} \vec{PQ} \|$ is it

Ex: Distance of $10x + 2y - 2z = 5$ to origin $(0,0,0)$ Q

1. $P(\frac{1}{2}, 0, 0)$ $\quad \vec{n} = \langle 10, 2, -2 \rangle$
2. $\vec{PQ} = \langle -\frac{1}{2}, 0, 0 \rangle$
3. $\text{proj}_{\vec{n}} \vec{PQ} = \left| \frac{\langle -\frac{1}{2}, 0, 0 \rangle \cdot \langle 10, 2, -2 \rangle}{\sqrt{108}} \right| = \frac{5}{\sqrt{108}}$ ✓

1'. $R = \langle 0, 0, -\frac{5}{2} \rangle$
2'. $\vec{RQ} = \langle 0, 0, \frac{5}{2} \rangle$
3'. $\frac{\langle 0, 0, \frac{5}{2} \rangle \cdot \langle 10, 2, -2 \rangle}{\sqrt{108}} = \frac{5}{\sqrt{108}}$ ✓

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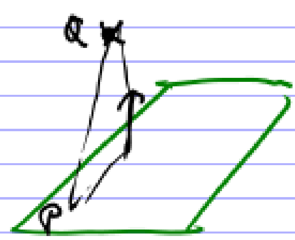
Panel 4

Find distance between

a) $10x + 2y - 2z = 5$ and $Q(1,1,1)$

b) $10x + 2y - 2z = 5$ and $x + y + z = 1$

c) $10x + 2y - 2z = 5$ and $5x + y - z = 1$



a) 1. $P(0, \frac{5}{2}, 0)$ is on plane

2. $\vec{PQ} = \langle 1-0, 1-\frac{5}{2}, 1-0 \rangle =$
$= \langle 1, -\frac{3}{2}, 1 \rangle$

$$d = \left| \frac{\vec{PQ} \cdot \vec{n}}{\|\vec{n}\|} \right| = \left| \frac{\langle 1, -\frac{3}{2}, 1 \rangle \cdot \langle 10, 2, -2 \rangle}{\sqrt{108}} \right| =$$

$$= \frac{5}{\sqrt{108}}$$

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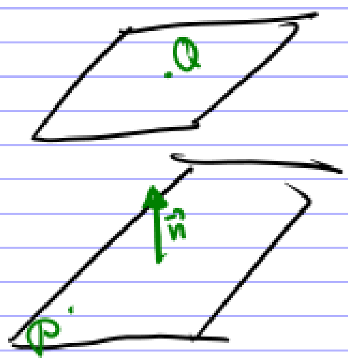
Panel 5

b) $10x + 2y - 2z = 5$ and $x + y + z = 1$

c) $10x + 2y - 2z = 5$ and $\boxed{5x + y - z = 1}$

b) The two planes are not parallel. Dist $= 0$

c) $\langle 10, 2, -2 \rangle = 2\langle 5, 1, -1 \rangle$ they are parallel. Thus:



Q any point on plane 1: $\langle \frac{1}{2}, 0, 0 \rangle$

P any point on plane 2: $\langle 0, 1, 0 \rangle$

$PQ = \langle \frac{1}{2}, -1, 0 \rangle$

$$d = \frac{\langle \frac{1}{2}, -1, 0 \rangle \cdot \langle 5, 1, -1 \rangle}{\sqrt{27}} = \frac{3/2}{\sqrt{27}}$$

Panel 6

We know the planes $10x + 2y - 2z = 6$ and $x + y + z = 1$ are not parallel. Thus, they intersect! Find intersection!

Dir. of line of intersection is perp. to both planes,

i.e. $\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 10 & 2 & -2 \\ 1 & 1 & 1 \end{vmatrix} = \langle 4, -12, 8 \rangle$

$\Rightarrow \vec{v} = \langle 1, -3, 2 \rangle$

line will pass through yz-plane (most likely), i.e. at that point $x = 0$.

$2y - 2z = 6$

$y + z = 1 \quad | \cdot 2$

$\Rightarrow 4y = 8, \; y = 2, \; z = -1$

$\ell(t) = \langle 0, 2, -1 \rangle + t\langle 1, -3, 2 \rangle$

$\ell(t) = \langle 1, -1, 1 \rangle + t\langle 1, -3, 2 \rangle$

Panel 7

Thm: Distance between $P(x_1, y_1, z_1)$ and $ax+by+cz+d=0$ is

$$D = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

Ex: dist. of origin $(0,0,0)$ to $10x+2y-2z-5=0$

is: $\frac{|10\cdot 0+2\cdot 0-2\cdot 0-5|}{\sqrt{108}} = \frac{5}{\sqrt{108}}$

Proof: 1) $Q(x_0, y_0, z_0)$ on plane

2) $PQ = (x_0-x_1, y_0-y_1, z_0-z_1)$

3) $d = \left|\frac{\vec{PQ}\cdot\vec{n}}{\|\vec{n}\|}\right| = \frac{|\langle x_0-x_1, y_0-y_1, z_0-z_1\rangle\cdot(a,b,c)|}{\sqrt{a^2+b^2+c^2}}$

$= \frac{|a(x_0-x_1)+b(y_0-y_1)+c(z_0-z_1)|}{\sqrt{a^2+b^2+c^2}} = \frac{|\overbrace{ax_0+by_0+cz_0}^{-d}-ax_1-by_1-cz_1|}{\sqrt{a^2+b^2+c^2}}$

$= \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

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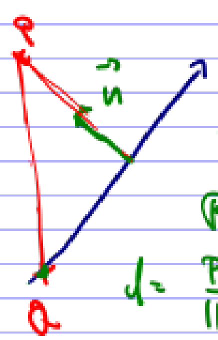
Panel 8

Find a formula for the distance between $P(x_0, y_0)$ and a line $ax+by+c=0$ (in 2D)



$\Rightarrow \vec{n} = \langle a, b\rangle$ is normal to line

1. $P(x_0, y_0)$, $Q(0, -\frac{c}{b})$

$PQ = \langle -x_0, -\frac{c}{b} - y_0\rangle$

$d = \frac{PQ\cdot n}{\|n\|} = \frac{\langle -x_0, -\frac{c}{b}-y_0\rangle\langle a,b\rangle}{\sqrt{a^2+b^2}} = \frac{|-ax_0-c-by_0|}{\sqrt{a^2+b^2}} = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$

Direction of line:

$(0, -\frac{c}{b})$

$(-\frac{c}{a}, 0)$ are on the line

Which vector is $\perp$ to $(b,-a)$

$(a,b)\cdot(b,-a)=0$

$\Rightarrow$ dir. vector is $\langle 0-(-\frac{c}{a}), -\frac{c}{b}-0\rangle = \langle \frac{c}{a}, -\frac{c}{b}\rangle \left(\frac{ab}{c}\right) = \langle b, -a\rangle$

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Panel 9

Distance between

$P_0(x_0, y_0) \in \mathbb{R}^2$ and line $ax+by+c=0$ in $\mathbb{R}^2$

$$d = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$$

$P_0(x_0, y_0, z_0) \in \mathbb{R}^3$ and plane $ax+by+cz+d=0$ in $\mathbb{R}^3$

$$d = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

Distance between: 2 planes ✓
1 plane, 1 line ✓
2 lines in $\mathbb{R}^2$ ✓ 2 lines in $\mathbb{R}^3$!

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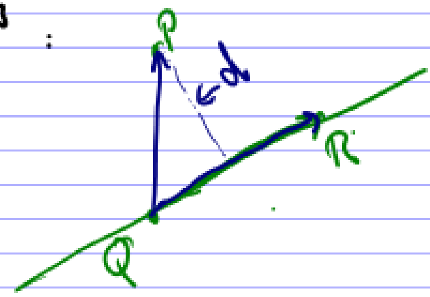
Panel 10

Distance of point $P(x_0, y_0)$ from line $ax+by+c=0$ in $\mathbb{R}^2$:

$$d = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$$

Distance of point $P(x_0, y_0, z_0)$ from line through R and Q in $\mathbb{R}^3$:

$$d = \frac{\|\vec{QR} \times \vec{QP}\|}{\|QR\|}$$



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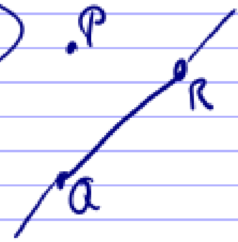
Panel 11

Ex: 4) $\ell(t) = \langle 1,2,3\rangle + t\langle -1,-2,-1\rangle$ and $P(0,0,2)$

Note: $\ell(1) = P$ i.e. distance is zero

5) $\ell(t)$ and $P(1,5,-3)$ .P



$$d = \frac{\|QR \times QP\|}{\|QR\|}$$

$Q(1,2,3)$, $R(0,0,2) \Rightarrow QR = \langle -1,-2,-1\rangle$

$QP = \langle 0,3,-6\rangle$

$$\|\langle -1,-2,-1\rangle \times \langle 0,3,-6\rangle\| = \begin{vmatrix} i & j & k \\ -1 & -2 & -1 \\ 0 & 3 & -6 \end{vmatrix} \quad (15,-6,-3)$$

$$d = \sqrt{225+36+9} \Big/ \sqrt{1^2+2^2+1^2} = \#$$

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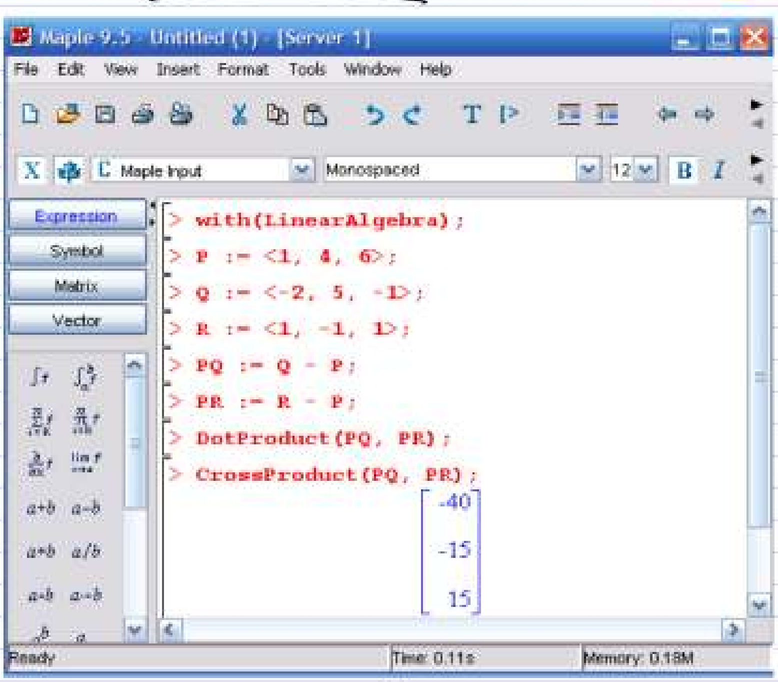
Panel 12

Maple: Dot + Cross Product



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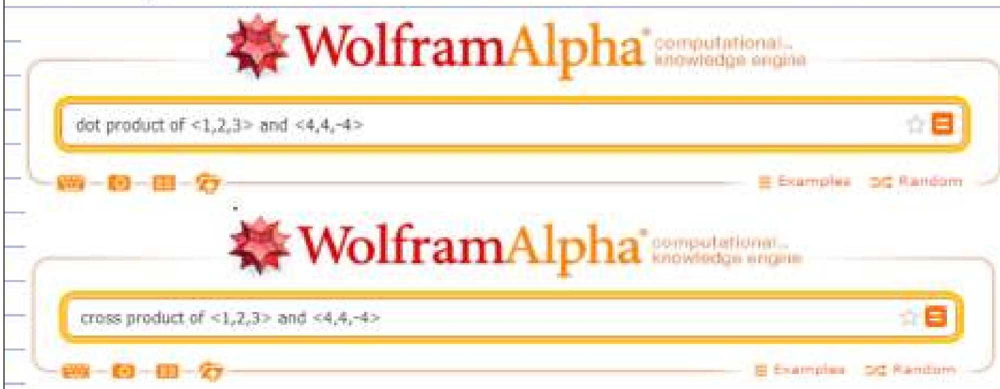
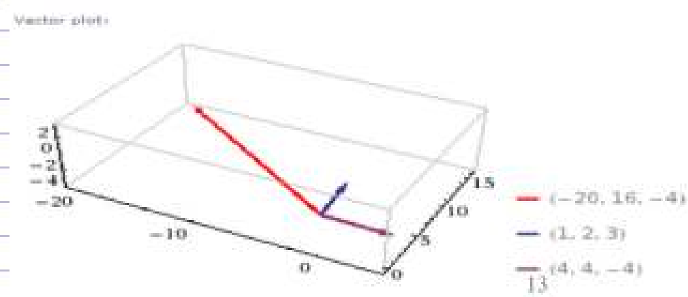
Panel 13

Wolfram Alpha Dot / Cross Products





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