This is a practice exam. The actual exam consists of questions of the type found in this practice exam, but will be shorter. If you have questions do not hesitate to send me email. Answers will be posted if possible – no guarantee.

1. Definitions: Please state in your own words the following definitions:
   a) Limit of a function $z = f(x, y)$
   b) Continuity of a function $z = f(x, y)$
   c) Partial derivative of a function $f(x, y)$
   d) Gradient and its properties
   e) Directional derivative of a function $f(x, y)$ in the direction of a unit vector $u$
   f) The (definition and geometric meaning of) the double integral of $f$ over the region $R \iint_R f(x, y) \, dA$
   g) Surface area

2. Theorems: Describe, in your own words, the following:
   a) A theorem relating differentiability with continuity
   b) The procedure to find relative extrema of a function $f(x, y)$
   c) The procedure to find absolute extrema of a function $f(x, y)$
   d) How to switch a double integral to polar coordinates
   e) A theorem that allows you to evaluate a double integral easily

3. True/False questions:
   a) If $\lim_{(x,y) \to (0,0)} f(x,y) = 0$ then $\lim_{x \to 0} f(x,0) = 0$ True. If general limit exists, the more specific one also does.
   b) If $\lim_{y \to 0} f(y,0) = 0$ then $\lim_{(x,y) \to (0,0)} f(x,y) = 0$ False. If limit exists only in a direction, it is not possible for general limit.
   c) $\lim_{n \to 0} \frac{f(x+ah,y+bh)-f(x,y)}{n} = \frac{\partial}{\partial x} f(x,y)$ True. This is the definition of $\frac{\partial}{\partial x} f(x,y)$.
   d) If $f$ is continuous at $(0,0)$, and $f(0,0) = 10$, then $\lim_{(x,y) \to (0,0)} f(x,y) = 10$ True. By the very definition of continuity.
   e) If $f(x,y)$ is continuous, it must be differentiable. False. $f(x,y) = |x|$.
   f) If $f(x,y)$ is differentiable, it must be continuous. True by theorem.
   g) If $f(x,y)$ is a function such that all second order partials exist and are continuous then $f_{xx} = f_{yy}$ False. $f_{xy} = f_{yx}$ but not for $f_{xx}$ and $f_{yy}$.
   h) The volume under $f(x,y)$, where $a \leq x \leq b$ and $g(x) \leq y \leq h(x)$ is $\int_a^b \int_{g(x)}^{h(x)} f(x,y) \, dy \, dx$ False. The correct expression is $\int_a^b \int_{g(x)}^{h(x)} f(x,y) \, dy \, dx$.
   i) If $f(x,y)$ is continuous then $\int_a^b \int_c^d f(x,y) \, dx \, dy = \int_c^d \int_a^b f(x,y) \, dy \, dx$ True. Fubini's Theorem.
If $f(x,y)$ is continuous then
\[
\iint_{R} f(x,y) \, dx \, dy = \left( \int_{a}^{b} f(x) \, dx \right) \left( \int_{c}^{d} g(y) \, dy \right)
\]

If $f$ is continuous over a region $D$ then \[\int_{D} f(x,y) \, dx \, dy = \int_{D} f(r,\theta) \, r \, dr \, d\theta \]

4. Surfaces: Find the domain for the following functions
   a) \[f(x,y) = \frac{1}{xy}, \quad \text{domain: } xy \neq 0 \]
   b) \[f(x,y) = \frac{1}{x^2 + y}, \quad \text{domain: } (x,y) \neq (0,0)
   \]
   c) \[f(x,y) = \frac{1}{x^2 - y^2}, \quad \text{domain: } x^2 - y^2 \neq 0 \]

5. Limits and Continuity: Determine the following limits as $(x,y) \rightarrow (0,0)$, if they exist.
   a) \[\lim_{(x,y) \rightarrow (0,0)} \frac{xy + 1}{x^2 + y^2 + 1} = 1\]
   b) \[\lim_{(x,y) \rightarrow (0,0)} \frac{xy + 1}{x^2 + y^2} \text{ undefined}
   \]
   c) \[\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} = 0 \]
   d) \[\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + y^2}
\]

Note: \[\frac{x^2}{x^2 + y^2} < 1 \rightarrow \left| \frac{x^2 y}{x^2 + y^2} \right| < 1 \rightarrow \left| y \right| < \sqrt{x^2 y^2} \]

Then: Given $\varepsilon > 0$ pick $\delta = \varepsilon$. Then if
\[|x^2 y^2| < \delta \rightarrow \sqrt{x^2 y^2} < \delta \rightarrow \varepsilon
\]
6. **Picture:** Match the following contour plots (level plots) to their corresponding surfaces.

- **d)** [1] \[ \approx \]
- **[2] \approx \]
- **e)** [3] \[ \approx \]
- **[4] \approx \]
- **f)** [A] \[ \approx \]
- **[B] \approx \]
- **g)** [C] \[ \approx \]
- **[D] \approx \]

**Other picture problems:**
- Given a contour plot, draw the gradient vector at specific points
- Classify some regions as type-1, type-2, or neither.
7. **Differentiation**: Find the indicated derivatives for the given function:

a) Find

b) Suppose \( f(x, y) = 2x^3 y^2 + 2y + 4x \), find

\[
\begin{align*}
    f_x &= 6x^2 y^2 + 4 \\
    f_y &= 4x^3 y + 2 \\
    f_{xx} &= 12x y^2 \\
    f_{xy} &= 12x^2 y \\
    f_{yy} &= 4x^3 \\
    f_{yx} &= 12x^2 y \\end{align*}
\]

(c) Find the rate of change with respect to \( y \) of \( x^2 + y^2 + z^2 = 1 \) at \( P\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)\)

\( \text{at } P \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) \)

\[
\begin{align*}
    f_{xy} &= 2x^2 y \\
    f_{yy} &= 2e^x \\
    f_{yx} &= 2e^x \\
    f_{yx} &= 2e^x \\
    \text{match by coincidence} \\
    f_{yxx} &= 2e^x \\
\end{align*}
\]

\( \text{match by coincidence} \)
8. **Directional Derivatives:**

a) Find the directional derivative of \( f(x, y) = xy e^{xy} \) at \((-2, 0)\) in the direction of a vector \( u \), where \( u \) makes an angle \( \pi/4 \) with the x-axis.

\[
\nabla f(x, y) \cdot u = f_x u_x + f_y u_y = xy e^{xy} \cos(\pi/4) + ye^{xy} \sin(\pi/4) \quad \text{at } (-2, 0) \Rightarrow f_x = 0, f_y = -2
\]

\[\langle x, y \rangle \cdot \langle \cos(\pi/4), \sin(\pi/4) \rangle \cdot C = (\sqrt{2}, \sqrt{2}) \]

b) Find \( D_u(f) \) where \( f(x, y) = \frac{x}{y} - \frac{y}{x} \) and \( \vec{u} = \langle -\frac{4}{5}, \frac{3}{5} \rangle \)

\[
f_x = \frac{1}{y} - \frac{y}{x^2} \quad \Rightarrow \quad D_u(f) \left[ \begin{pmatrix} \frac{1}{y} - \frac{y}{x^2} \end{pmatrix} \right] = \left( \frac{1}{y} - \frac{y}{x^2} \right) \cdot \left( \frac{4}{5}, \frac{3}{5} \right) = \left( \frac{4}{5} \left( \frac{1}{y} - \frac{y}{x^2} \right) - \frac{3}{5} \left( \frac{1}{y} - \frac{y}{x^2} \right) \right)
\]

c) Suppose \( f(x, y) = x^2 e^y \). Find the maximum value of the directional derivative at \((-2, 0)\) and compute a unit vector in that direction.

\[
\text{Max dir. deriv. at } (-2, 0) \Rightarrow
\cases{f_x = 2x e^y \quad \Rightarrow \quad D_u(f) \left[ \begin{pmatrix} 2x e^y \end{pmatrix} \right] \quad \neq \quad \| \nabla f \| = \| \sqrt{(4x)^2 + y^2} \| = \sqrt{16} \\
\}
\]

10. **Max/Min Problems:** Compute the extrema as indicated

a) \( f(x, y) = 3x^2 - 2xy + y^2 - 8y \). Find relative extreme and saddle point(s), if any.

\[
\begin{align*}
f_x &= 6x - 2y = 0 \\
f_y &= -2x + 2y - 8 = 0
\end{align*}
\]

\[\Rightarrow \quad x = 2, y = 6 \quad \text{is a critical point.}
\]

\[\nabla \cdot \begin{pmatrix} 6 & -2 \\ -2 & 2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \begin{pmatrix} 6 \\ -2 \end{pmatrix} \end{pmatrix} \text{ is a minimum at } (-2, 4) \quad \text{and } f_{xx} = 6 > 0 \]

b) \( f(x, y) = 4xy - x^4 - y^4 \). Find relative extrema and saddle point(s), if any

\[
\begin{align*}
f_x &= 4y - 4x^3 = 0 \quad \Rightarrow \quad y = x^3 \\
f_y &= 4x - 4y^3 = 0 \quad \Rightarrow \quad x = y^3
\end{align*}
\]

\[\begin{cases}x = 0, y = 0 \quad \Rightarrow \quad 3 \text{ critical points} \\
x = 1, y = 1 \\
x = -1, y = -1
\end{cases}
\]

\[\nabla \cdot \begin{pmatrix} -12x^3 & 4 \\ 4 & -12y^3 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} -12 \\ 4 \end{pmatrix} \text{ is a saddle point at } (0,0)
\]

\[\begin{pmatrix} -12 & 4 \\ 4 & -12 \end{pmatrix} = 144x^2 y^2 - 16 \]

\[\begin{cases}x = 0, y = 0 \Rightarrow \text{max at } (1,1) \\
x = 0, f_{xx} < 0 \Rightarrow \text{max at } (1,1) \\
y = 0, f_{yy} < 0 \Rightarrow \text{max at } (-1,-1)
\end{cases}
\]
c) Let \( f(x, y) = 3xy - 6x - 3y + 7 \). Find absolute maximum and minimum inside the triangular region spanned by the points \((0,0), (3, 0), \) and \((0, 5)\).

\[
\begin{align*}
G_x &= 3y - 6 = 0, \quad G_y = 3x - 3 = 0 \Rightarrow (1, 2) \text{ critical points} \\
\frac{\partial f}{\partial x} &= 3y - 6, \quad \frac{\partial f}{\partial y} = 3x - 3 \\
\frac{\partial^2 f}{\partial x^2} &= 0, \quad \frac{\partial^2 f}{\partial y^2} = 0, \quad \frac{\partial^2 f}{\partial x \partial y} = 3 \\
G_{xx} &= \frac{\partial^2 f}{\partial x^2} = 0, \quad G_{yy} = \frac{\partial^2 f}{\partial y^2} = 0, \quad G_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 3 \\
G_{xx} G_{yy} - G_{xy}^2 &= -9 < 0 \\
\therefore (1, 2) &\text{ is a saddle point} \\
\end{align*}
\]

Let \( f(x, y) = 3x^2 - 2xy + y^2 - 8y \). Find the absolute extrema over \([0, 1] \times [0, 2]\).

\[
\begin{align*}
x=0: & \quad y^2 - 8y \rightarrow y^2 - 8y + 16 \rightarrow y \leq 4 \\
x=1: & \quad 3y^2 + y - 8y \rightarrow -5y + 1 \rightarrow y \leq 2 \\
x=0: & \quad 3y^2 - 2xy + y^2 - 8y \\
y=0: & \quad 3x^2 \rightarrow x \leq 0 \\
y=2: & \quad 3x^2 - 2x(2) + 4 \rightarrow 4x - 4 \rightarrow x = \frac{3}{2} \\
\therefore \text{max} &\text{ at } (0, 4) \quad \text{and} \quad (\frac{3}{2}, 2) \\
\text{min} &\text{ at } (0, 2) \\
\end{align*}
\]

11. Evaluate the following integrals:

a) \[
\int_0^1 \int_0^x y^2 \, dy \, dx = \int_0^1 \frac{x^3}{3} \, dx = \frac{1}{3} \left[ x^3 \right]_0^1 = \frac{1}{3}
\]

b) \[
\int_0^{\pi/2} \int_0^\pi \sin(x) \cos(y) \, dy \, dx = \int_0^\pi \left[ \sin(y) \cos(x) \right]_0^{\pi/2} \, dx = \int_0^\pi \sin(x) \, dx = \left[ -\cos(x) \right]_0^\pi = 2
\]

c) \[
\int_0^1 \int_{x^2}^1 \frac{1}{y} x \, dy \, dx = \int_0^1 \left[ \frac{1}{y} x \right]_{x^2}^1 \, dx = \int_0^1 \left( \frac{1}{x^2} - 1 \right) x \, dx = \int_0^1 \left( 1 - x^3 \right) x \, dx = \int_0^1 \left( x - x^4 \right) \, dx = \left[ \frac{1}{2} x^2 - \frac{1}{5} x^5 \right]_0^1 = \frac{3}{10}
\]

d) \[
\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} r^2 \sin^3 \theta \, dr \, d\theta = \int_0^{\pi} \int_0^2 r^2 \sin^3 \theta \, dr \, d\theta = \int_0^{\pi} \frac{1}{3} r^3 \sin^3 \theta \bigg|_0^2 \, d\theta = \frac{1}{3} \left[ \frac{1}{4} r^4 \right]_0^2 = \frac{8}{3}
\]
\[ \iiint_{R} z^2 + y^2 + z^2 \, dx \, dy \, dz = \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} z^2 + y^2 + z^2 \, dx \, dy \, dz = \int_{0}^{1} \int_{0}^{1} \left( \frac{1}{3} (z^3 + y^3 + z^3) \right) \, dy \, dz \]
\[ = \frac{1}{3} (z^3 + y^3 + z^3) \bigg|_{0}^{1} \]
\[ = \frac{1}{3} (1 + 1 + 1) = \frac{3}{3} = 1 \]

\[ \int_{0}^{\pi/2} \int_{0}^{1} \cos(x^2) \, dy \, dx = \int_{0}^{1} \int_{0}^{\pi/2} \cos(x^2) \, dx \, dy = \frac{1}{2} \sin(x^2) \bigg|_{0}^{1} = \frac{1}{2} (\sin(1) - \sin(0)) = 0 \]

\[ \iint_{R} \sqrt{x^2 + y^2} \, dA \] where \( R \) is the part of the circle in the 1st quadrant
\[ = \int_{\theta = 0}^{\pi/2} \int_{r = 0}^{1} r \sqrt{r^2} \, dr \, d\theta = \int_{\theta = 0}^{\pi/2} \int_{r = 0}^{1} r^2 \, dr \, d\theta \]
\[ = \frac{1}{3} r^3 \bigg|_{0}^{1} \int_{\theta = 0}^{\pi/2} d\theta = \frac{1}{3} \left( \frac{\pi}{2} \right) = \frac{\pi}{6} \]

12. The pictures below show different ways that a region \( R \) in the plane can be covered. Which picture corresponds to the integral \( \iint_{R} f(x, y) \, dx \, dy \)?

\[ \text{This is right} \]
\[ \text{This is right} \]
\[ \text{This is right} \]

\[ \text{This is right} \]
13. Suppose you want to evaluate $\int \int_R f(x,y) \, dA$ where $R$ is the region in the $xy$ plane bounded by $y=0$, $y=2-x^2$, and $y=x$. According to Fubini’s theorem you could use either the iterated integral $\int \int f(x,y) \, dy \, dx$ or $\int \int f(x,y) \, dx \, dy$ to evaluate the double integral. Which version do you prefer? Explain. You do not need to actually work out the integrals.

14. Use a multiple integral and a convenient coordinate system to find the volume of the solid:
   a) bounded by $z=x^2-y+4$, $x=0$, $y=0$, and $x=4$
   b) bounded by $z=e^{-x^2}$ and the planes $y=0$, $y=x$, and $x=1$
   c) bounded above by $z=\sqrt{16-x^2-y^2}$ and bounded below by the circle $x^2+y^2 \leq 4$
   d) evaluate $\int \int \frac{y}{x^2+y^2} \, dx \, dy$ where $R$ is a triangle bounded by $y=x$, $y=2x$, and $x=2$
   e) bounded by the paraboloid $z=4-x^2-2y^2$ and the $xy$ plane
15. Find the following surface areas:

a) of the plane $z = 2 - x - y$ above the rectangle $0 \leq x \leq 2$ and $0 \leq y \leq 3$

$$
\int_0^2 \int_0^3 \sqrt{1 + (-1)^2 + (-1)^2} \, dy \, dx = \int_0^2 \int_0^3 \sqrt{3} \, dy \, dx
$$

b) of the cylinder $z = 9 - x^2$ above the triangle bounded by $y = x$, $y = -x$, and $y = 3$

$$
\int_{-3}^3 \int_{-x}^3 \sqrt{1 + (-2x)^2 + 0} \, dy \, dx = \int_{-3}^3 \int_{-x}^3 \sqrt{1 + 4x^2} \, dy \, dx
$$

c) of the surface $z = 16 - x^2 - y^2$ above the circle $x^2 + y^2 \leq 9$

$$
\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \sqrt{1 + (-2x)^2 + (-2y)^2} \, dy \, dx
$$

16. Prove the following facts:

a) Use the definition to find $f_x$ for $f(x, y) = xy$

$$
f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} = \lim_{h \to 0} \frac{(x+h)y - xy}{h} = \lim_{h \to 0} y = y
$$

b) Use the definition to find $f_y$ for $f(x, y) = xy$

$c)$ A function $f$ is said to satisfy the Laplace equation if $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$. Show that the function $f(x, y) = \ln(x^2 + y^2)$ satisfies the Laplace equation.

$$
f_x = \frac{2x}{x^2 + y^2}, \quad f_y = \frac{2y}{x^2 + y^2}
$$

$$
f_{xx} = \frac{2(x^2 - x^2)}{(x^2 + y^2)^2} = \frac{0}{(x^2 + y^2)^2},
\quad f_{yy} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}
$$

So, $f_{xx} + f_{yy} = \frac{0}{(x^2 + y^2)^2} + \frac{2(x^2 - (x^2 - y^2))}{(x^2 + y^2)^2} = 0$. Therefore, $f(x, y) = \ln(x^2 + y^2)$ satisfies the Laplace equation.
d) Two functions $u(x, y)$ and $v(x, y)$ are said to satisfy the Cauchy-Riemann equations if

\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \]

Show that the functions $u(x, y) = e^x \cos(y)$ and $v(x, y) = e^x \sin(y)$ satisfy the Cauchy-Riemann equations.

\[
\begin{align*}
u_x &= e^x \cos(y) \\
u_y &= e^x \sin(y) \\
u_x &= -e^x \sin(y) \\
u_y &= e^x \cos(y)
\end{align*}
\]

So $u_x = v_y$ and $u_y = -v_x$.

e) Let $f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{for } (x, y) \neq (0, 0) \\
0 & \text{for } (x, y) = (0, 0) \end{cases}$ Then show that $f$ has partial derivatives at $(0, 0)$ but $f$ is not differentiable at $(0, 0)$ – hard!

f) Prove that the volume of a sphere with radius $R$ is $\frac{4}{3} \pi R^3$.

\[
\begin{align*}
V &= \iiint_{R^3} 1 \, dx \, dy \, dz \\
&= 2 \iint_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sqrt{R^2-x^2-y^2} \, dy \, dx \\
&= 2 \int_{-R}^{R} \left[ -\frac{1}{3} (R^2-x^2)^{3/2} \right]_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \, dx \\
&= \frac{4}{3} \pi R^3
\end{align*}
\]

g) Prove that the surface area of a sphere with radius $R$ is $4 \pi R^2$.

\[
\begin{align*}
A &= \iint_{S} \sqrt{\frac{x^2}{R^2-x^2} + \frac{y^2}{R^2-x^2} + 1} \, dx \, dy \\
&= 2 \iiint_{S} \sqrt{\frac{x^2}{R^2-x^2} + \frac{y^2}{R^2-x^2}} \, dx \, dy \\
&= 2 \int_{0}^{R} \int_{0}^{\pi} \sqrt{1 - \frac{x^2}{R^2}} \, d\theta \, dx \\
&= 4 \pi R^2
\end{align*}
\]