Panel 1

Last Time

Function \( f(x,y) \) \[
\int_\gamma f(x,y) \, ds
\]
\[
\int_\gamma f(x,y) \, dx + \int_\gamma f(x,y) \, dy
\]

Vector field \( \mathbf{F} = (M,N) \) - \[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \, dA
\]

Fundamental Thm of Line Integration:

Panel 2

**Fundamental Theorem for Line Integrals**

If \( \mathbf{F} \) is conservative with potential function \( f \), and \( \gamma(t) \), \( a \leq t \leq b \), a smooth curve. Then:

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = f(\gamma(b)) - f(\gamma(a))
\]

Consequences. If \( \mathbf{F} \) is conservative then:

1. \[
\oint_{\gamma_1} \mathbf{F} \cdot d\mathbf{r} = \oint_{\gamma_2} \mathbf{F} \cdot d\mathbf{r} \quad \text{for any two curves with same start/end point}
\]

2. \[
\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \quad \text{for any closed curve} \ \gamma.
\]
Panel 3

\[ F(x,y) = \sqrt{x^2 + y^2} \] 

\[ F C \] parabola \( y = 2x^2 \) from \((-1,2) \) to \((2,8)\)

\[ \int_C \int 7 \, dx \, dy = \int x^2 \, dx + y^2 \, dy = \]

\[ \int_{-1}^{2} t^3 \, dt + \left(2t^4\right)^2 \, dt = \int t^2 + 16t \, dt - \frac{1}{3} t^3 + \frac{16}{6} t^2 \bigg|_{-1}^{2} = \]

\[ \frac{2}{3} t^2 + 16t - \frac{1}{3} - \frac{1}{6} = \frac{2}{3} t^2 + 16 + 1 - \frac{8}{3} \]

\[ \Rightarrow \int_C x^2 \, dx + y^2 \, dy = \frac{1}{3} (x^2 + y^2) \bigg|_{(-1,0)}^{(2,0)} = \frac{1}{3} (8 + 4) - (\frac{1}{3}) - \frac{1}{6} = \]

\[ \frac{8 + 24 + 1 - 2}{3} = \frac{33}{3} \]

Panel 4

\[ F = (y^2, x^2, xy + z^2) \]

\[ F_x = y^2 \quad \Rightarrow \quad F = xy + C(y, z) \]

\[ F_y = x^2 + C(y, z) = x^2 \quad \Rightarrow C_y(y, z) = 0 \quad \Rightarrow C(y, z) = z + C \]

\[ F_z = xy + C(z) \]

\[ F_0 = xy + C(z) = xy + 2z \quad \Rightarrow \quad C(z) = z^2 + C \]

\[ f(x, y, z) = xy + z^2 \]
Panel 5

\[ \text{Ex:} \quad \text{Let } F(x, y) = \left( \frac{y^2}{1 + x^2}, 2y \arctan(x) \right) \text{ and } \]
\[ r(t) = \left( t^2, 2t \right), \quad t \in [0, 1]. \text{ Find } \int_0^1 F(r(t)) \, dt \]

\[ \int_0^1 F(r(t)) \, dt = \int_0^1 \frac{y^2}{1 + x^2} \, dx + 2y \arctan(x) \, dy = \frac{1}{2} \left( \left. \frac{d(1)}{dt} - \frac{d(0)}{dt} \right) = 4 \arctan(1) \]

\[ \text{Where } f_x = \frac{y^2}{1 + x^2} \Rightarrow f = y^2 \cdot \arctan(x) \]

Panel 6

\[ \text{Ex:} \quad \text{Find } \int \tan(y) \, dx + x \sec^2(y) \, dy = 0 \]

where \( f(t) = \left( \cos(t), \sin(t) \right), \quad t \in [0, 2\pi] \]

because \( f \) is closed and \( \mathbb{F} \) is conservative.
Panel 7

**Summary of Conservative Vector Field**

\[ \mathbf{F} = \nabla f, \quad f \text{ is potential function} \]

\[ \int_C \mathbf{F} \cdot d\mathbf{r} \text{ is independent of path from } A \text{ to } B \]

\[ \int_C \frac{\partial \mathbf{F}}{\partial z} dz = 0 \text{ for all closed curves } C \]

Thus, if domain is simply connected and all potials are continuous, then

\[ \text{curl}(\mathbf{F}) = 0 \quad \Rightarrow \quad \mathbf{F} \text{ conservative} \]

Note: Simply connected means "no holes"

Panel 8

Find a conservative vector field that has the given potential:

\[ f(x, y, z) = \sin(x^2 + y^2 + z^2) \]

Find \( \text{div}(\nabla \cdot F) \) and \( \text{curl}(F) = \nabla \times F \)

\[ F(x, y, z) = \langle x^2, y^2, y, y + 2z \rangle \]

Evaluate \( \int_C (x - y)dx + xdy \) if \( C \) is the graph of \( y^2 = x \) from \((4,-2)\) to \((4,2)\)

Find the work done by \( F(x, y, z) \) along the curve \( \langle t, t^2, t^3 \rangle \) from \((0, 0, 0)\) to \((2, 4, 8)\), where \( F(x, y, z) = \langle y, z, x \rangle \)

Check which of the following vector fields is not conservative.

\[ F(x, y) = \langle 3x^2 + 2, x + 4y \rangle \]

\[ F(x, y) = \langle e^x, 3 - e^y \sin(y) \rangle \]

\[ F(x, y, z) = \langle 8xz, 1 - 6yz, 4x^2 - 9y^2z \rangle \]

Show that the line integrals are independent of the path, and find their value:

\[ \int_C (y^3 + 2xy)dx + (x^2 + 2xy)dy \]

\[ \int_C (6xy^2 + 2x^2)dz + (9x^2y^2)dx + (4xz + 1)dy \]
Panel 9

\[ \int_{\text{line int}} \Rightarrow \text{Green's Theorem} \]

Panel 10

Green's Theorem. If \( R \) is a region in \( xy \)-plane with boundary curve \( C \). \( C \) is piecewise smooth, non-intersecting, and positively oriented. \( \vec{F} = (M, N) \) is a smooth vector field. Then,

\[
\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA
\]

pos. oriented: as you walk along the curve, interior in to the field!
Panel 11

**Example 1:** Evaluate \( \oint_C xy \, dx + x^3 \, dy \), where \( C \) is as shown.

**Method A:** Green's Theorem

\[
\int_C xy \, dx + x^3 \, dy = \iint_R (3x^2 - 3x) \, dA = \\
= \int_0^2 \int_x^2 (3x^2 - 3x) \, dy \, dx = \\
= \int_0^2 3x^2 - 3x \, dx \, \bigg|_x^2 = \frac{22}{3}
\]

**Method B:** \( \int_0^2 (x^2 + y^2) \, dx + (x^2 + y^2) \, dy = \# \)

\[
\int_0^2 (x^2 + y^2) \, dx + (x^2 + y^2) \, dy = \# \implies \frac{2y}{15}
\]

Panel 12

**Example 2:** Evaluate \( \oint_C 2xy \, dx + (x^2 + y^2) \, dy \), where \( C \) is \( 4x^2 + 9y^2 = 36 \).

Think (Green's)

\[
\iint_R (\nabla \cdot F) \, dA = 0
\]

(\( F \) is conservative)
Panel 13

Ex: Find
\[ \oint_C \left( x \sin(y^2) - y \right) \, dx + \left( x^2 y \cos(y^2) + 3x \right) \, dy \]

where \( C \) is the triangle \((0,0), (1,0), (0,1)\).

by Green: \[ \iint_R \left( 2xy \cos(y^2) + 1 - 2y x \cos(y^2) - 1 \right) \, dA \]
\[ = \int_C 2 \, dA = 2 \int_0^\pi \int_0^1 r \, dr \, d\theta = \frac{2 \pi}{2} \]
\[ = \pi \]

Panel 14

Evaluate
\[ \oint_C \left( 3y - e^\sin(x) \right) \, dx + \left( x + \sqrt{y^2 + 1} \right) \, dy \]

where \( C \) is the circle \( x^2 + y^2 = 9 \).

\[ \oint_C -4 \, dA = 3 \oint_C \, d\theta = 3 \cdot \pi \cdot (3)^2 \]
\[ = 27\pi \]
Theorem: If \( D \) is a region enclosed by a curve \( C \), then
\[
\text{area}(D) = \frac{1}{2} \int_C x \, dy - y \, dx.
\]

\[
\int_a^b \int_{f(x)}^{g(x)} x \, dy - y \, dx = \int_a^b \left( \frac{1}{2}(-1) \right) \, dx = \int_a^b dx = 2 \text{ area}(D).
\]

Example:
Find area enclosed by \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).

The region is an ellipse centered at \((0,0)\) with semi-major axis \(a\) and semi-minor axis \(b\).

\[
\text{area} = \frac{1}{2} \int_a^b x \, dy - y \, dx = \frac{1}{2} \int_a^b a \cos(t) \, b \sin(t) - b \sin(t) \, a \cos(t) \, dt = \frac{1}{2} \int_a^b a \cos(t) \, b \sin(t) \, dt = \frac{1}{2} \int_a^b a \sin(2t) \, dt = \frac{1}{2} a b \int_a^b \sin(2t) \, dt = \frac{1}{4} a b \left[ \cos(2t) \right]_a^b = \frac{1}{4} a b \left( -2 \right) = \frac{1}{2} a b.
\]
Ex: Evaluate \( \int_C y^2 \, dx + 3xy \, dy \) where \( C \) is the boundary of the region between \( x^2 + y^2 = 1 \) and \( x^2 + y^2 = 4 \), \( \frac{1}{2} \) upper half plane.

\[
\int_C y^2 \, dx + 3xy \, dy = \int_{C_1} + \int_{C_2}
\]

\[
\int_0^{\pi/2} \int_0^2 3y - 2y \, r \, dr \, d\theta = \int_0^{\pi/2} \int_0^2 r \sin(\theta) \, r \, dr \, d\theta
\]

15. Evaluate \( \int_C (2x + y) \, dx + (2x + y) \, dy \), \( C \) curve from \((-2, 2)\) to \((4, 3)\).

16. Find the work done by the force field \( F = <9x^2y^2, 6x^3y - 1> \) from \( P(0,0) \) to \( Q(5,9) \).
18. Evaluate $\int_C 2xydx + (x + y)dy$ where $C$ bounds the region between $y = 0$ and $y = 4 - x^2$.

21. Evaluate $\int_C x\sin(y^2) - y^2)dx + (x^2\cos(y^2) + 3x)dy$ where $C$ is the boundary of the trapezoid with vertices $(0, -2), (1, -1), (1, 1), \text{ and } (0, 2)$. 