**Panel 1**

**Summary**

\[ \vec{r}(t) = \text{space curve} \]

\[ \vec{r}'(t) = \text{tangent} \]

\[ \vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \quad \text{unit tangent} \]

\[ \vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} \quad \text{unit normal} \]

\[ \vec{B}(t) = \vec{T}(t) \times \vec{N}(t) \quad \text{binormal} \]

\[ \chi(t) = \frac{\|\vec{T}(t)\|}{\|\vec{r}(t)\|} = \frac{\|\vec{T}(t) \times \vec{N}(t)\|}{\|\vec{r}(t)\|^3} \quad \text{curvature} \]

**Panel 2**

**Example**

Let \[ \vec{r}(t) = (\cos(t), \sin(t), t) \]. Find tangent, unit normal and binormal vectors at \( t = 0 \)

\[ \vec{T}(t) = \frac{1}{\sqrt{2}} \left< -\sin(t), \cos(t), 1 \right> \]

\[ \vec{N}(t) = \left< -\cos(t), -\sin(t), 0 \right> \]

\[ \vec{B}(t) = \frac{1}{\sqrt{2}} \left< \sin(t), -\cos(t), 1 \right> \]

At \( t = 0 \):

\[ \vec{T}(0) = \left( 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \]

\[ \vec{N}(0) = \left< -1, 0, 0 \right> \]

\[ \vec{B}(0) = \left< 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right> \]
**Panel 3**

**Def:** The plane determined by $T$ and $N$ is called the osculating plane or *supporting plane*.

**Def:** The circle in the osculating plane with radius $r = \frac{1}{\kappa}$ is called the osculating circle.

**Panel 4**

**Ex:** Find the osculating plane of $\mathbf{r}(t) = (\cos(t), \sin(t), t)$ at $\mathbf{P}(0, 1, \frac{\pi}{2})$.

We know:

$T(t) = \frac{1}{\sqrt{1 + \frac{1}{4} \left( -\sin(t), \cos(t), 1 \right)}^2}$

$N(t) = \left( -\cos(t), -\sin(t), 0 \right)$

$\mathbf{B}(t) = \frac{1}{\sqrt{1 + \frac{1}{4} \left( -\sin(t), -\cos(t), 1 \right)}^2}$

$\mathbf{B}(\frac{g}{2}) = \left( 0, 0, 1 \right)$ or $\mathbf{B}(1, 0, 1)$

$x + b = 0$
Panel 5

Ex: Find the osculating circle to \( y = x^2 \) at \((0,0)\).

Clearly for a 2D graph, the osc. plane is \( z=0 \)

\[ r^2(t^2 < 1, 1^2, 0) \]
\[ r'(1) < (1, 2\hat{t}, 0), r'(0) < (1, 0, 0) \]
\[ r''(1) < (0, 2, 0), r''(0) < (0, 2, 0) \]

\[ r' \times r'' = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{vmatrix} \Rightarrow c < 0, 0, 1^2 \]

\[ X = \frac{1}{r'(1)^2} \Rightarrow \text{radius is } \frac{1}{4} \]

\[ (x)^2 + (y - \frac{1}{2})^2 = \frac{1}{4} \]

Panel 6

Motion in Space

Suppose \( \vec{r}(t) \) represents the motion or path of a particle through space or time:

\[ \vec{r}(t) = \text{motion in space} \]

\[ \vec{v}(t) = \text{velocity} \quad \vec{v}(t) = \dot{\vec{r}}(t) \]

\[ \left| \vec{v}(t) \right| = \text{speed} \quad \left| \vec{v}(t) \right| = \left| \vec{r}'(t) \right| \]

\[ \vec{a}(t) = \text{acceleration} \quad \vec{a}(t) = \vec{v}'(t) = \ddot{\vec{r}}(t) \]
Panel 7

Ex: Suppose the path of a particle at time \( t \) is
\[ \mathbf{r}(t) = < t^3, t^2 >. \]
Find velocity, speed, and acceleration when \( t = 1 \). Illustrate.

Panel 8

Ex: A particle starts at \( P(1,0,0) \) with initial velocity \( <1,-1,1> \). The acceleration is \( \mathbf{a}(t) = <4t,6t,1> \).
Find velocity, speed, and position.
Panel 9

Ex: An object with mass $m$ moves in a circle with constant angular speed $\omega$. Find the force acting on the object and illustrate.

$r(t) = <\cos(\omega t), \sin(\omega t)> \text{ circle}$

$r'(t) = <-\omega \sin(\omega t), \omega \cos(\omega t)> \text{, } s = \omega t$

$r''(t) = <-\omega^2 \cos(\omega t), -\omega^2 \sin(\omega t)>$

$\mathbf{F} = m \mathbf{a} = -m \omega^2 r(t)$.

$\mathbf{F}$ points towards origin (always!)


Panel 10

Application of Motion

A baseball is hit 3 feet above ground at 100 feet per second and at an angle of $\pi/4$ with respect to the ground. Find the maximum height reached by the baseball. Will it clear a 10-foot high fence located 300 feet from home base?

$r(0) = (0, 3)$

$v(0) = \frac{100}{\sqrt{2}} (1, 1)$

$a(t) = <0, -g>$

$v(t) = <x(t) - 9.8 t + c_1, y(t) - \frac{100}{\sqrt{2}} t + c_2>$

$a(t) = \left< \frac{100}{\sqrt{2}}, -9.8 t + \frac{100}{\sqrt{2}} \right>$

$q(t) = \left< \frac{100}{\sqrt{2}} t + d_1, y(t) - \frac{1}{2} 9.8 t^2 + \frac{100}{\sqrt{2}} t + d_2 \right>^2$
Tangential and Normal Components of Acceleration

acceleration can be divided

- portion in direction of \( \mathbf{T} \): course change in speed
- portion in direction of \( \mathbf{N} \): course change in direction

\[
\dot{a} = a_T \mathbf{T} + a_N \mathbf{N}
\]

- \( a_T \) in tangential comp. of \( \dot{a} \)
- \( a_N \) is normal

Panel 12

**Theorem:** \( \dot{a} = a_T \mathbf{T} + a_N \mathbf{N} \) where

- tangential component \( a_T = \frac{v \cdot a}{s} \)
- normal component \( a_N = \frac{\parallel v \times a \parallel}{s} \)

**Example:** \( r(t) = \langle t^2, t^2, t^3 \rangle \) - find \( a_T \) and \( a_N \)

\( r'(t) = \langle 2t, 2t, 3t^2 \rangle \) \( r''(t) = \langle 2, 2, 6t \rangle \) \( s = \sqrt{8t^2 + 9t^3} \)

\( a_T = \frac{\dot{r} \cdot \dot{r} \times \ddot{r}}{\| \dot{r} \times \ddot{r} \|} = \frac{1}{\sqrt{8t^2 + 9t^3}} \left[ \begin{array}{c} 1 \\ 2t \\ 2 \end{array} \right] \)

\( a_N = \)
Quiz 4

Suppose $\mathbf{r}(t) = \langle t^3, 2, t \rangle$ is a vector-valued function (aka space curve), representing the position of a particle. Find the following:

1. The velocity at $P(0,0,0)$
2. The speed at $P(0,0,0)$
3. The acceleration at $P(0,0,0)$
4. The unit tangent $\mathbf{T}(t)$ at $P(0,0,0)$
5. The unit normal vector $\mathbf{N}(t)$ at $P(0,0,0)$
6. The bi-normal vector $\mathbf{B}(t)$ at $P(0,0,0)$
7. The curvature $\kappa$ at $P(0,0,0)$
8. The tangential component of the acceleration $a_T$ at $P(0,0,0)$
9. The normal component of the acceleration $a_N$ at $P(0,0,0)$
10. The osculating plane at $P(0,0,0)$
11. The osculating circle at $P(0,0,0)$