Panel 1

Conservative Vector Fields

Conservative \( F \) \( \Rightarrow \) \( \nabla \cdot F = 0 \) \( \Rightarrow \) \( \text{curl}(F) = 0 \) \( \Rightarrow \) \( \frac{\partial N}{\partial y} = \frac{\partial M}{\partial x} \)

Fundamental Theorem of Line Integration: \( F \) conservative

\[ \int_C \nabla \phi \cdot dl = \phi(B) - \phi(A) \]

\( \int_C \nabla \phi \cdot dl \)

\( \int_C \nabla \phi \cdot dl = 0 \)

Panel 2

Find a conservative vector field that has the given potential:
\( f(x, y, z) = \sin(x^2 + y^2 + z^2) \)

Find \( \text{div}(\nabla \cdot F) \) and \( \text{curl}(F) = \nabla \times F \)
\( F(x, y, z) = \langle x^2z, y^2x, x + 2z \rangle \)

Evaluate \( \int_C (x - y)dx + xdy \) if \( C \) is the graph of \( y^2 = x \) from (4, -2) to (4, 2)

Find the work done by \( F(x, y, z) \) along the curve \( \langle t, t^2, t^3 \rangle \) from (0, 0, 0) to (2, 4, 8), where
\( F(x, y, z) = y, y, x \)

Check which of the following vector fields is not conservative.
\( F(x, y) = \langle 3x^2y + 2x^3 + 4y^3 \rangle \)
\( F(x, y) = \langle e^x, 3 - e^x \sin(y) \rangle \)
\( F(x, y, z) = \langle 8xz, y^2, 4x^3 - 9y^2z^3 \rangle \)

Show that the line integrals are independent of the path, and find their value:
\[ \int_{(1, 1)}^{} (y^3 + 2xy)dx + (x^3 + 2xy)dy \]
\[ \int_{(-2, 1)}^{} (6xy^3 + 2x^2)dx + (9x^2y^2)dy + (4xz + 1)dz \]

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Panel 3

\[
\int_C \nabla \cdot \mathbf{F} \, dx = \int_R \nabla \times \mathbf{F} \cdot d\mathbf{A} = \int_R \left( \frac{\partial N}{\partial y} - \frac{\partial M}{\partial x} \right) \, dA
\]

Panel 4

**Green's Theorem:** Let \( R \) be a region in \( xy \)-plane with a piecewise smooth, non-intersecting, and positively oriented boundary curve \( C \). \( \mathbf{F} = (M, N) \) is a smooth vector field. Then:

\[
\int_C \mathbf{F} \cdot d\mathbf{C} = \int_R \left( \frac{\partial N}{\partial y} - \frac{\partial M}{\partial x} \right) \, dA
\]

**Note:** If \( \mathbf{F} \) is conservative:

\[
\int_R \left( \frac{\partial N}{\partial y} - \frac{\partial M}{\partial x} \right) \, dA = 0
\]

2D Theorem: There is a 2D version - Cauchy
Panel 5

Ex: Evaluate \( \int_C xy \, dx + x^3 \, dy \), where \( C \) is as shown.

Method 1: old way

\[
\int_0^2 \int_0^{x^2} \left( xy \ dx + x^3 \ dy \right) = \int_0^2 \int_0^{x^2} 2x \ dy \ dx + \int_0^2 \int_0^{x^2} 4x \ dy \ dx + \int_0^2 \int_0^{x^2} 8x^3 \ dy \ dx = -\frac{40}{15}
\]

Method 2: Green's Theorem

\[
\int_C xy \, dx + x^3 \, dy = \int_R \left( \frac{\partial}{\partial y} (x^3) - \frac{\partial}{\partial x} (xy) \right) \, dA = \int_R \left( 3x^2 - x \right) \, dA = \int_0^2 \int_0^{x^2} 3x^2 - x \, dy \ dx = \int_0^2 \left[ \frac{3x^3}{3} - \frac{x^2}{2} \right]_{y=0}^{y=x^2} \, dx = -\frac{40}{15}
\]

Panel 6

Ex: Evaluate \( \int_C 2xy \, dx + (x^2 + y^2) \, dy \), \( C \) is \( 4x^2 + 9y^2 = 36 \)

Old way: \( r(t) = \left< \frac{2}{3} \cos(t), \frac{2}{3} \sin(t) \right> \) maybe...?

Green's Theorem:

\[
\int_C 2xy \, dx + (x^2 + y^2) \, dy = \iint_R 2x - 2x \ dA = 0
\]

Way funnier
Panel 7

\[ \text{Ex: Find } \int_{\gamma} M \, dx + N \, dy \]

where \( \gamma \) is the triangle \((0,0), (1,0), (0,1)\).

Old way: 3 integrals

New way: \( \int_{\gamma} (2y \cos y^2 + 3 - 2x \cos y^2 - 1) \, dA = \int_{\gamma} 2 \, dA = \int_{\gamma} 2 \cdot \text{area}(\gamma) = 2 \cdot \frac{1}{2} = 1 \)

Panel 8

Evaluate \[ \int_{C} M \, dx + N \, dy \]

where \( C \) is the circle \( x^2 + y^2 = 9 \).

Old way: one integral, and \( x(t) = 3 \cos t, \quad y(t) = 3 \sin t \)

Green's Theorem: \( \int_{\gamma} (7 - 3) \, dA = 4 \int_{\gamma} dA = 4 \cdot 9 = 36 \pi \)
Theorem: If $D$ is a region enclosed by a curve $C$ then \[
\text{area}(D) = \frac{1}{2} \oint_C x \, dy - y \, dx \]

Proof: \[
\begin{align*}
\oint_C x \, dy - y \, dx &= \iint_D \left( \frac{\partial}{\partial x} \frac{\partial}{\partial y} \right) \, dx \, dy \\
&= - \iint_D 1 - (-1) \, dx \, dy \\
&= 2 \int_0^1 \int_{x_1}^{x_2} dx \, dy = 2 \text{ area}(D)
\end{align*}
\]

Panel 9

Example: Find area enclosed by \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \)

Area of ellipse \( r(t) = \langle a \cos(t), b \sin(t) \rangle \), \( t \in [0, 2\pi] \)

\[ \begin{align*}
x(t) &= a \cos(t) \\
y(t) &= b \sin(t) \\
A &= \frac{1}{2} \oint_C x \, dy - y \, dx \\
&= \frac{1}{2} \int_0^{2\pi} a^2 \cos^2(t) dt + b^2 \sin^2(t) dt \\
&= \frac{1}{2} \int_0^{2\pi} (a^2 + b^2) dt \\
&= \frac{1}{2} (2\pi a^2 + 2\pi b^2) \\
&= \pi (a^2 + b^2)
\end{align*} \]

Area of circle: \( A = \pi r^2 \) \( \text{circum. of ellipse} = \frac{1}{2} (a + b) \)

Length of circle: \( L = \int_0^{2\pi} \sqrt{r'(t)^2 + (r(t))^2} \, dt \)

\[
\begin{align*}
r(t) &= \langle r \cos(t), r \sin(t) \rangle \\
L &= \pi r \approx \frac{2\pi R}{4} \end{align*}
\]
**Panel 11**

Review: Volume of sphere radius $R$?

$$x^2 + y^2 + z^2 = R^2 \Rightarrow z = \sqrt{R^2 - x^2 - y^2}$$

$$V = \iiint \sqrt{R^2 - x^2 - y^2} \, dV = \iiint_R \sqrt{R^2 - x^2 - y^2} \, dy \, dx$$

$$= 2\pi \int_0^R \left[ \frac{1}{3} \left( R^2 - r^2 \right)^{3/2} \right]_0^R - \frac{2}{3} \pi \left( -R^2 \right)$$

$$= \frac{4}{3} \pi R^3$$

**Panel 12**

More Review: Surface area of a sphere radius $R$?

$$f(x, y) = \sqrt{R^2 - x^2 - y^2}$$

$$S = \iint_S \, dS = \iint_R \sqrt{1 + \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2} \, dx \, dy$$

$$f_x = \frac{x}{\sqrt{R^2 - x^2 - y^2}}$$

$$f_y = \frac{y}{\sqrt{R^2 - x^2 - y^2}}$$

$$S = 4\pi R^2$$

<table>
<thead>
<tr>
<th>Volume</th>
<th>Surface Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^2$</td>
<td>$2\pi r^2$</td>
</tr>
<tr>
<td>$R^3$</td>
<td>$4\pi r^2$</td>
</tr>
</tbody>
</table>

Hint project!
Panel 13

Evaluate \( \int_C y^2 \, dx + 3xy \, dy \) where \( C \) is the boundary of the region between \( x^2 + y^2 = 1 \) and \( x^2 + y^2 = 4 \).

Green's Theorem applies.

\[
\begin{align*}
\int_C y^2 \, dx + 3xy \, dy &= \int_{\Theta} 3y - 2y \, r \, dr \, d\theta \\
&= \int_0^1 \int_0^{2\pi} r \sin^2(\theta) \, r \, dr \, d\theta \\
&= 0
\end{align*}
\]

Panel 14

15. Evaluate \( \int_C 2(x + y) \, dx + 2(x + y) \, dy \), \( C \) curve from \((-2, 2)\) to \((4, 3)\).

Find potential \( \Rightarrow f(0) - f(2) \)

16. Find the work done by the force field \( F = <9x^2 y^2, 6x^3 y - 1> \) from \( P(0, 0) \) to \( Q(5,9) \).

\[
\int_{L} 9x^2 y^2 \, dx + 6x^3 y - 1 \, dy
\]

Find potential \( \Rightarrow f(5) - f(0) \)
18. Evaluate \( \int_C 2xydx + (x + y)dy \) where \( C \) bounds the region between \( y = 0 \) and \( y = 4 - x^2 \).

Long way or Green's

21. Evaluate \( \int_C \sin(y^2) - y^2)dx + (x^2 \cos(y^2) + 3x)dy \) where \( C \) is the boundary of the trapezoid with vertices (0, -2), (1, -1), (1, 1), and (0, 2).

Long way = \( \neq \) integrals

Green's Then