Panel 1

Last Time

Chain Rule: \[ f(x, y), x = x(t), y = y(t) \]
\[ \frac{df}{dt} = \frac{df}{dx} \frac{dx}{dt} + \frac{df}{dy} \frac{dy}{dt} \]

Directional Derivative: \[ D_u f = \nabla f \cdot \hat{u} \]

Gradient: \[ \nabla f = (f_x, f_y) \]

Properties of Gradient: see next slide

Panel 2

Properties of Gradient

- The gradient is a \textit{vector}
- Gradient is \textit{perpendicular} to level curves
- Gradient points in direction of \textit{max. increase}
- \[ \| \nabla f \| \text{ is the max. rate of change} \]

Ex: Find \( \nabla f \) if \( f(x, y) = \ln(x y^2 + 5) \)

\[ \nabla f = (f_x, f_y, f_z) = \left( \frac{y^2 e^2}{x y^2 + 5}, \frac{2 x y e^2}{x y^2 + 5}, \frac{3 x y^2}{x y^2 + 5} \right) \]

\[ = \left( \frac{1}{x}, \frac{2}{y}, \frac{3}{z} \right) \]
Panel 3

Ex: Suppose the level curves of an area are given by
\( f(x,y) = y \ln(x) \). You are standing at \( P(1,3) \)
and you are heading in the direction \(-4,3\).
Are you going up or down? How much?

Compute \( D_\alpha f (\xi) \) at \( P(1,3) \)

\[
D_\alpha f (\xi) \cdot \nu = \left< \frac{\partial f}{\partial x}, \ln(x) \right> \frac{1}{x} < -4,3 \quad \text{at} \quad P(1,3): \\
\left< -3, 0 \right> \frac{1}{x} < -4,3 \right) = \frac{12}{x} > 0
\]

Thus it’s going up by \( \frac{12}{x} \) units.

Panel 4

Name: 

1. Consider the function \( f(x,y) = x^2 + 3xy - y^2 \). Find
   \( a) \ 6x \)
   \( b) \ \frac{\partial f}{\partial y} \)
   \( \frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial y} + \frac{\partial f}{\partial y} \cdot \frac{\partial f}{\partial x} = 0 \)
   \( c) \ \nabla f \)

\( x(0) = 0 \quad y(0) = 5 \)

2. If \( f(x,y) = x^2 + 3xy^2 \) and \( x = \sin(\theta), y = \cos(\theta) \). Find
   \( \frac{\partial f}{\partial \theta} \)

\( \frac{\partial f}{\partial \theta} = 2 \cos(\theta) + 6 \sin(\theta) = 2 \)

\( \frac{\partial f}{\partial \theta} = -3 \sin(\theta) - 2 \cos(\theta) = 0 \)

\( \frac{\partial f}{\partial \theta} = y + 3y^2 \quad \text{at} \quad (0,1) = 4 \)
Panel 5

3. \( f(x,y) = x^3 - 3xy + 4y^4 \). Find the directional derivative in the direction of \( \cos(\frac{\pi}{4}), \sin(\frac{\pi}{4}) \).

\[
D_u f = \lim_{t \to 0} \frac{f(u + tv) - f(u)}{t}, \quad u = \langle a, b \rangle
\]

\( \nabla f \cdot u \)

4. Consider the contour plot below. Sketch the gradient:

a) at \( P(1.5, 1.0) \)

b) at \( P(0.0) \)

Panel 6

Review of Max/Min problems in \( \mathbb{R} \)

Local max.  Local max.

\( f = 0 \)  \( f' = 0 \)  \( f'' = 0 \)

\( f'' > 0 \)  \( f'' < 0 \)  \( f'' = 0 \)

0  \( f' \)

0  \( f' = 0 \) (Critical)

0  check \( f'' \)
Panel 8

Max/Min Problems

To find max/min of \( z = f(x, y) \):

1. Find \( \nabla f \):
   \[
   \begin{align*}
   f_x &= f_{xx} \\
   f_y &= f_{xy}
   \end{align*}
   \]

2. Solve \( \nabla f = 0 \) (system of equations)

3. Compute
   \[
   H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}
   \]
   and \( D = f_{xx} f_{yy} - (f_{xy})^2 \)
   - Hessian matrix
   - \( a) \) if has min if: \( D > 0, f_{xx} > 0 \)
   - \( b) \) if has max if: \( D > 0, f_{xx} < 0 \)
   - \( c) \) if has saddle if: \( D < 0 \)
   - \( d) \) no information if: \( D = 0 \)
Panel 9

Ex: Find and classify the critical points for

\[ f(x, y) = x^2 - 2xy + 3y^2 + 4x \]

1. \( D_x : \quad f_x = 2x - 2y + 4 \]
   \[ f_y = -2x + 6y \]

2. \( D_y : \quad 2x - 2y + 4 = 0 \]
   \[ -2x + 6y = 0 \]
   \[ 4y^2 = 0 \quad \Rightarrow \quad y = 0, x = \frac{3}{2} \]
   \( x = \frac{3}{2} \) is a critical point.

3. \( \nabla^2 f = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 6 \end{pmatrix} : \quad D = 2 \cdot 6 - (-2)^2 = 12 - 4 = 8 > 0 \]
   \[ f_{xx} = 2 > 0 \]

Panel 10

\[ f(x, y) = x^2 - 2xy + 3y^2 + 4x \]
Panel 11

Suppose \( f(x,y) = x^2 + 2y^2 + 4xy \). Find and classify all relative extrema, if any.

\[
\begin{align*}
 f_x &= 2x + 4y = 0 \quad \Rightarrow \quad 4y = -2x \\ 
 f_y &= 8y + 4x = 0 \quad \Rightarrow \quad 2x - 8y = 0 \\

(0,0) &\text{ is critical} \\

H &= \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 4 & 4 \end{pmatrix} \\

D &= 8 - 16 = -8 < 0
\]

Saddle point

Panel 12

Find and classify critical points for \( f(x,y) = 3x - x^2 - 2y^2 \)

\[
\begin{align*}
 f_x &= 3 - 2x = 0 \quad \Rightarrow \quad x = 1, -1 \\
 f_{yy} &= -4y = 0 \quad \Rightarrow \quad y = 0 \\

(1,0) \text{ and } (-1,0)
\]

\[
H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} -6 & 0 \\ 0 & -4 \end{pmatrix} \Rightarrow D = 24 > 0
\]

Thus, \((1,0)\) : \(D > 0, f_{yy} < 0 \Rightarrow \text{max} \\
(-1,0) : D < 0 \Rightarrow \text{saddle}
Ex: Find and classify the critical points for \( f(x,y) = x^3y + 12x^2 - 8y \).

But guess from plot/contour plot a saddle but where? 