Sheets + Coordinate Systems

\[ x^2 + y^2 = 1 \] base around y-axis

**Vectors:**
- add, subtract, length, visually
- dot prod, cross product, angles
- projection

**Planes + Lines:**
- parametric or scalar equations, intersections, distances, angles

**Vector-valued functions:**
- limits, derivatives, integrals
- tangents, unit tangent, normal, binormal, length, curvature

Motion in Space:
- velocity, speed, accel., normal + tangential
- comp. of accel., slinky problem

**Formulas:**

\[ \theta = \cos^{-1}(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}) \]

\[ \mathbf{v} = \mathbf{r}'(t) \]

\[ v = \|\mathbf{r}'(t)\| \]

\[ s = \int \|\mathbf{r}'(t)\| \, dt \]

\[ T = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \]

\[ N = \frac{\mathbf{r}''(t)}{\|\mathbf{r}''(t)\|} \]

\[ \mathbf{B} = T \times N \]

\[ k = \frac{\|\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}''(t)\|^2}{\|\mathbf{r}'(t)\|^3} \]

Supporting plane: spanned by \( T \) and \( N \)

Osculating circle: circle in curve plane will

\[ r = \int \|\mathbf{r}'(t)\| \, dt \]
Application of Motion

A baseball is hit 3 feet above ground at 100 feet per second and at an angle of Pi/4 with respect to the ground. Find the maximum height reached by the baseball. Will it clear a 10-foot high fence located 300 feet from home base?

\[ r(t) = \]

\[ \begin{align*}
    a(t) &= (0, -g) \\
    v(t) &= (v_{x0}, v_{y0}) = (\frac{100}{\sqrt{2}}, -g + \frac{100}{\sqrt{2}}) \\
    x(t) &= (100/\sqrt{2})t + \frac{1}{2}gt^2 + (100/\sqrt{2})t + 3
\end{align*} \]

1. Max height:
   \[ y(t) = -\frac{1}{2}gt^2 + \frac{100}{\sqrt{2}}t + 3 \quad \text{max} \]
   \[ y(t = 0) = 3 \]
   \[ y(t) \text{ is max. height} \]

2. Does it clear a 10-foot wall at 300 feet?

\[ \text{Wall: } x = \frac{100}{\sqrt{2}}t = 300 \Rightarrow t = \frac{300 \cdot \sqrt{2}}{100} = 3\sqrt{2} \]

\[ \text{Height: } y(3\sqrt{2}) = -\frac{1}{2} \cdot 32 \cdot (3\sqrt{2})^2 + \frac{100}{\sqrt{2}} \cdot 3\sqrt{2} + 3 = 15 \]

\[ = -16 \cdot 18 + 300 + 3 \]

HR!
Panel 5

True/False questions

1. \( r(t) = \langle t^3, 2t^2, 3t^3 \rangle \) is a line. \(< u_1, u_2, u_3 > \) lies in it. \( \text{Yes} \)

\[ \frac{d}{dt} (v(t) \times w(t)) = v'(t) \times w(t) \] \( \text{No} \) my example

\[ \frac{d}{dt} \|r(t)\| = \|r'(t)\| \] \( \text{No} \) my example

\[ \|r(t)\| = 5 \] for all \( t \) \( \text{True} \)

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More about distances, planes, and vectors

Panel 6

Vectors: Suppose \( u = \langle 7, -2, 3 \rangle \), \( v = \langle -1, 4, 5 \rangle \), and \( w = \langle -2, 1, -3 \rangle \)

- Are \( u \) and \( v \) orthogonal, parallel, or neither?
- Find graphically and algebraically \( 2u + 3v \) and \( u - v \)
- Find the angle between \( v \) and \( w \)
- Find \( u \cdot v \) (dot product), \( u \times v \) (cross product), \( u \cdot (v \times w) \), and \( ||u|| \)

\[ 7 - 1 + 12 - 4 + 9 - 5 = -3 - 1 + 15 = 0 \]

Two vectors are parallel if \( c \mathbf{w} = \mathbf{v} \)

\[ x \mathbf{v} = \langle -1, 4, 5 \rangle = \langle -2, 1, -3 \rangle \]

\[ \mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) = 0 \]
Panel 7

Equation of line through \( P(1,2,3) \) and \( Q(4,5,1) \)

\[ \mathbf{L} = (1,2,3) + t(3,3,-2) = \mathbf{P} + t\mathbf{PQ} \]

Equation of plane through \( P(1,0,2), Q(2,1,3), R(1,1,1) \)

\[ \begin{align*}
    a(x-x_0) + b(y-y_0) + c(z-z_0) &= 0 \\
    \mathbf{a} \cdot \mathbf{n} &= 0 \\
    ax+by+cz+d &= 0
\end{align*} \]

Intersection of lines/planes

Panel 8

Distance

Find distance of \( P(3,1) \) to line \( 2x-y = 1 \)

\[ d = \frac{\mathbf{PQ} \cdot \mathbf{n}}{\|n\|} \quad \mathbf{n} = (2, -1) \quad \mathbf{Q} = (4, 1) \]

Find distance of \( P(1,0,3) \) to plane \( 2x + 5y + z = 0 \)

same as above.
Panel 9

If \( r(t) = \langle 4t, t^2, t^3 \rangle \), find \( r'(t) \), \( r''(t) \), \( \frac{d}{dt} \|r(t)\| \)

If \( r(t) = \langle e^t, 3t^3, \frac{3}{6t} \rangle \) some curve, find \( \int_1^2 r(t) \, dt \) / \( t \)

If \( r(t) = \langle t, \frac{1}{t} \rangle \), find \( T(t) \), \( N(t) \), \( a_t \) and \( a_n \)

If \( r(t) = \langle 3 + t, 2t, 1 - 4t \rangle \), find \( N \). Explain.

\( r(\frac{1}{4}) \in \langle 4, \frac{1}{16}, \frac{1}{64} \rangle \), \( \|r\| = \sqrt{1 + \left(\frac{1}{4}\right)^2} \sqrt{1 + \left(\frac{1}{16}\right)^2} \)

\( T = \frac{1}{\sqrt{1 + \frac{1}{16}}} \langle 1, -\frac{1}{4} \rangle \Rightarrow N = \frac{1}{\sqrt{\frac{1}{16} + 1}} \langle \frac{1}{4}, 1 \rangle \)

Panel 10

If \( r(t) = \langle 3 - 8t, 4t, t \rangle \) find length of curve as \( t \in [0, 1] \)

\[ s = \int_0^1 \|r'(t)\| \, dt = \int_0^1 \sqrt{\langle 3 - 8t, 4t, t \rangle, \langle 3 - 8t, 4t, t \rangle} \, dt = \int_0^1 \sqrt{6} \, dt = \sqrt{6} \]

be prepared for in class and quiz.
Panel 11

Find curvature of \( \mathbf{r}(t) = <t, t^2, \frac{t^3}{6}> \) at \( t = 1 \)

\[ \chi = \frac{\| \mathbf{r}' \|}{\| \mathbf{r}' \times \mathbf{r}'' \|} \]

\[ \mathbf{r}'(t) = <1, 2t, \frac{t^2}{2}> \quad \Rightarrow \quad \mathbf{r}'(1) = <1, 2, \frac{1}{2}> \]

\[ \mathbf{r}''(t) = <0, 2, \frac{t}{2}> \quad \Rightarrow \quad \mathbf{r}''(1) = <0, 2, \frac{1}{2}> \]

Panel 12

Picture Problems:

7. Picture: Sketch the circle that fits the graph below the best at the points \( x = 0 \) and \( x = 3 \). At which of the two points is the curvature smaller?
Panel 13

Panel 14

Panel 13

Picture Problems

Sketch $T$, $N$, $u$, $u_t$, and $u_N$ at $t = 3$

Panel 14

Picture Problems

Match graphs to functions

sinkers, spirals, horseshoe, etc.

1. B

2. A

3. C

$0 \ x(t) < \cos(\theta(t))$ $0 \ y^2 + x^2 = 1$
Panel 15

Story Problem

What is the maximum height and range of a projectile fired at a height of 3 feet above the ground with an initial velocity of 900 feet/sec and at an angle of 45 degrees above the horizontal?

Panel 16

Proof:

Prove the following facts:
1. Show that $u \times v = -(v \times u)$
2. Show that $u \cdot (v \times u) = 0$
3. Show that if $y = f(x)$ is a function that is twice continuously differentiable, then the curvature of $f$ at a point $x$ is $K = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}}$
4. Prove that the curvature of a line in space is zero.

$y = \beta(x) \Rightarrow v(t) = <t, \beta(t), 0>$

$\frac{\|v'(x) \times v''(x)\|}{\|v'(x)\|^3}$