Panel 1

Last Time we discussed vector-valued functions

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$. We talked about:

Tangent vectors: $\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$

Unit tangent vectors: $\vec{T}(t) = \vec{r}\,'(t) / \|\vec{r}\,'(t)\|$

Length: $s = \int_a^b \sqrt{(f')^2 + (g')^2 + (h')^2}\,dt = \int_a^b \|\vec{r}\,'(t)\|\,dt$

Curvature: $\kappa = \dfrac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|}$

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Panel 2

Ex: Find the curvature for $\vec{r}(t) = \langle t, t^2 \rangle$

$\kappa = \dfrac{\|T'(t)\|}{\|r'(t)\|}$ so we need r' and T first:

$r(t) = \langle t, t^2 \rangle \Rightarrow r'(t) = \langle 1, 2t \rangle \Rightarrow \|r'(t)\| = \sqrt{1+4t^2}$

Thus $T(t) = r'(t) / \|r'(t)\| = \dfrac{1}{\sqrt{1+4t^2}} \langle 1, 2t \rangle$

Note that T almost always involves a square root.

Now: $T'(t) = \dfrac{d}{dt}\left((1+4t^2)^{-1/2} \langle 1, 2t \rangle \right) =$ product rule

chain rule

$= -\frac{1}{2}(1+4t^2)^{-3/2} \cdot 8t \langle 1, 2t \rangle + (1+4t^2)^{-1/2} \langle 0, 2 \rangle =$

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Panel 3

$$= \frac{-4t}{(1+4t^2)^{3/2}} \langle 1, 2t \rangle + \frac{1}{(1+4t^2)^{1/2}} \langle 0, 2 \rangle =$$

factor ... add components

$$= \frac{1}{(1+4t^2)^{1/2}} \left[\left\langle -\frac{4t}{(1+4t^2)} + 0, \frac{-8t^2}{(1+4t^2)} + 2 \right\rangle \right] =$$

LCD

$$= \frac{1}{(1+4t^2)^{1/2}} \left\langle \frac{-4t}{(1+4t^2)}, \frac{-8t^2 + 2 + 8t^2}{(1+4t^2)} \right\rangle =$$

factor

$$= \frac{1}{(1+4t^2)^{3/2}} \langle -4t, 2 \rangle$$

$$\Rightarrow \|T'(t)\| = \frac{1}{(1+4t^2)^{3/2}} \|\langle -4t, 2 \rangle\| = \frac{1}{(1+4t^2)^{3/2}} \sqrt{4+16t^2} =$$

$$= \frac{2\sqrt{1+4t^2}}{(1+4t^2)^{3/2}} = \frac{2}{1+4t^2}$$

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Panel 4

So far we have:

$r(t) = \langle t, t^2 \rangle$, $r'(t) = \langle 1, 2t \rangle$, $\|r'(t)\| = \sqrt{1+4t^2}$

$$T(t) = \frac{1}{\sqrt{1+4t^2}} \langle 1, 2t \rangle$$

$$T'(t) = \frac{1}{(1+4t^2)^{3/2}} \langle -4t, 2 \rangle$$

$$\|T'(t)\| = 2/(1+4t^2)$$

Therefore the curvature works out to be:

$$\kappa = \frac{\|T'(t)\|}{\|r'(t)\|} = \frac{2}{1+4t^2} \cdot \frac{1}{\sqrt{1+4t^2}} = 2/(1+4t^2)^{3/2}$$

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Panel 5

Finding the curvature requires finding the derivative T' of the unit tangent T. Since T usually involves square roots, it is almost always painful to find T' and therefore the curvature. There is, however, a short cut:

Theorem: If $r: \mathbb{R}^3 \to \mathbb{R}$ is a 3D vector-valued function, the curvature κ can be found,

$$\kappa = \frac{\| r'(t) \times r''(t) \|}{\| r'(t) \|^3}$$

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Panel 6

In our previous example $r(t) = \langle t, t^2 \rangle$ is only a function in $\mathbb{R}^2$ so we can't apply the theorem. But, we could simply add a zero as third component:

$$\vec{r}(t) = \langle t, t^2 \rangle = \langle t, t^2, 0 \rangle$$

But then $\qquad \kappa = \dfrac{\| r'(t) \times r''(t) \|}{\| r'(t) \|^2}$

$r'(t) = \langle 1, 2t, 0 \rangle$

$r''(t) = \langle 0, 2, 0 \rangle$

$$\Rightarrow r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} = \langle 0, 0, 2 \rangle$$

$$\Rightarrow \underline{\underline{\kappa}} = \frac{\| r' \times r'' \|}{\| r' \|^3} = \frac{\| \langle 0, 0, 2 \rangle \|}{\left(\sqrt{1+4t^2}\right)^3} = \underline{\underline{\frac{2}{(1+4t^2)^{3/2}}}}$$

This is much easier!

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Panel 7

To continue we first need a theoretical result:

<u>Thm</u>: If $r(t)$ is a vector-valued function such that $\|r(t)\| \equiv 1$ $\forall t$ then $r(t)$ is perp. to $r'(t)$.

<u>Proof</u>: To show two vectors are perpendicular we need to show that their dot product is zero.

$$\Rightarrow 0 = \frac{d}{dt}(1) = \frac{d}{dt}\|r(t)\| = \frac{d}{dt}\, r(t)\cdot r(t) =$$

$$= r'(t)\cdot r(t) + r(t)\, r'(t) = 2\, r'(t)\cdot r(t)$$

$\Rightarrow$ $r(t)\cdot r'(t) = 0$ so r and r' are perpendicular. #

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Panel 8

Since for the unit tangent $T(t)$ we have $\|T\| = 1$ we know that T and T' are perpendicular to each other.

<u>Ex</u>: For $\vec{r}(t) = \langle t, t^2 \rangle$ we had:

$$T(t) = \frac{1}{\sqrt{1+4t^2}}\langle 1, 2t\rangle \quad , \quad T'(t) = \frac{1}{(1+4t^2)^{3/2}}\langle -4t, 2\rangle$$

$$\text{Thus } T\cdot T' = \frac{1}{(1+4t^2)^2}\langle 1, 2t\rangle\cdot\langle -4t, 2\rangle = 0$$

We normalize the vector T' and give it a new name:

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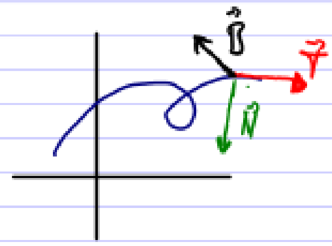
Panel 9

Def: If $T(t)$ is the unit tangent to a space curve $r(t)$ then $T'(t)$ is perpendicular to $T(t)$. We define:



$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$ to be the principle normal

$$B(t) = T(t) \times N(t)$$ to be the binormal

Note: At each point of a space curve $\vec{r}(t)$ the three vectors $\vec{T}, \vec{N}, \vec{B}$ are perpendicular to each other.

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Panel 10

Ex: Let $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$. Find tangent, unit normal and binormal vectors.

$r(t) = \langle \cos(t), \sin(t), t \rangle \Rightarrow r'(t) = \langle -\sin(t), \cos(t), 1 \rangle$

$\Rightarrow \underline{T(t)} = \frac{1}{\sqrt{2}} \underline{\langle -\sin(t), \cos(t), 1 \rangle}$ because $\|r'\| = \sqrt{2}$

$\Rightarrow T'(t) = \frac{1}{\sqrt{2}} \langle -\cos(t), -\sin(t), 0 \rangle$ and $\|T'\| = \frac{1}{\sqrt{2}}$

$\Rightarrow \underline{N(t)} = T' / \|T'\| = \underline{\langle -\cos(t), -\sin(t), 0 \rangle}$ so that

$$\vec{B} = \vec{T} \times \vec{N} = \begin{array}{c} \begin{array}{ccc} i & j & k \end{array} \\ \begin{vmatrix} \frac{-1}{\sqrt{2}}\sin(t) & \frac{1}{\sqrt{2}}\cos(t) & \frac{1}{\sqrt{2}} \\ -\cos(t) & -\sin(t) & 0 \end{vmatrix} \end{array} = \underline{\langle \tfrac{1}{\sqrt{2}}\sin(t), -\tfrac{1}{\sqrt{2}}\cos(t), \tfrac{1}{\sqrt{2}} \rangle}$$

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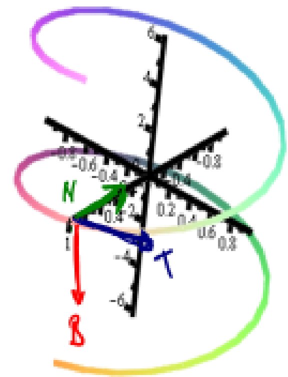
Panel 11

<u>Ex:</u> Let $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$. Find tangent, unit normal and binormal vectors at $t=0$

$T(t) = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle$

$N(t) = \langle -\cos(t), -\sin(t), 0 \rangle$

$B(t) = \frac{1}{\sqrt{2}} \langle \sin(t), -\cos(t), 1 \rangle$



$\underline{t=0}$:

$\Rightarrow \quad T = 1/\sqrt{2} \langle 0, 1, 1 \rangle$

$\Rightarrow \quad N = \langle -1, 0, 0 \rangle$

$\Rightarrow \quad B = \frac{1}{\sqrt{2}} \langle 0, -1, 1 \rangle$

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Panel 12

<u>Summary</u>

$r(t) = \langle f(t), g(t), h(t) \rangle$ — space curve

$r'(t) = \langle f'(t), g'(t), h'(t) \rangle$ — tangent vector

$T(t) = r'(t) / \|r'(t)\|$ — unit tangent

$N(t) = T'(t) / \|T'(t)\|$ — principle normal

$B(t) = T(t) \times B(t)$ — binormal

$\kappa = \dfrac{\|T'(t)\|}{\|r'(t)\|} = \dfrac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$ — curvature

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Panel 13

Quiz 4

Suppose $\vec{r}(t) = \langle t^2, 2, t \rangle$ is a vector-valued function (aka space curve), representing the position of a particle. Find the following:

1. The velocity at $P(0,0,0)$
2. The speed at $P(0,0,0)$
3. The acceleration at $P(0,0,0)$
4. The unit tangent $\vec{T}(t)$ at $P(0,0,0)$
5. The unit normal vector $\vec{N}(t)$ at $P(0,0,0)$
6. The bi-normal vector $\vec{B}(t)$ at $P(0,0,0)$
7. The curvature k at $P(0,0,0)$

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