

Math 2411 – Calc III Practice Exam 2

This is a practice exam. The actual exam consists of questions of the type found in this practice exam, but will be shorter. If you have questions do not hesitate to send me email. Answers will be posted if possible – no guarantee.

1. **Definitions:** Please state in your own words the following definitions:

- a) Limit of a function $z = f(x, y)$
- b) Continuity of a function $z = f(x, y)$
- c) partial derivative of a function $f(x, y)$
- d) gradient and its properties
- e) directional derivative of a function $f(x, y)$ in the direction of a unit vector u
- f) total differential
- g) The (definition and geometric meaning of) the double integral of f over the region R $\iint_R f(x, y) dA$
- h) Surface area

2. **Theorems:** Describe, in your own words, the following:

- a) a theorem relating differentiability with continuity
- b) a theorem stating criteria for a function to have relative extrema
- c) a result that classifies critical points into relative max., min., or saddle points
- d) the procedure to find relative extrema of a function $f(x, y)$
- e) the procedure to find absolute extrema of a function $f(x, y)$
- f) how to switch a double integral to polar coordinates
- g) a theorem that allows you to evaluate a double integral easily
- h) the “change of variables” theorem to change from rectangular to polar coordinates

3. **True/False** questions:

- a) If $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$ then $\lim_{x \rightarrow 0} f(x, 0) = 0$
- b) If $\lim_{y \rightarrow 0} f(0, y) = 0$ then $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$
- c) $\lim_{h \rightarrow 0} \frac{f(x+ah, y+bh) - f(x, y)}{h} = \frac{\partial}{\partial x} f(x, y)$
- d) If f is continuous at $(0, 0)$, and $f(0, 0) = 10$, then $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 10$
- e) If $f(x, y)$ is continuous, it must be differentiable
- f) If $f(x, y)$ is differentiable, it must be continuous
- g) If $f(x, y)$ is a function such that all second order partials exist and are continuous then $f_{xx} = f_{yy}$

- h) The volume under $f(x, y)$, where $a \leq x \leq b$ and $g(x) \leq y \leq h(x)$ is $\int_a^b \int_{g(x)}^{h(x)} f(x, y) dx dy$

- i) If $f(x, y)$ is continuous then $\int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$

- j) If $f(x, y)$ is continuous then $\int_a^b \int_c^d f(x)g(y) dy dx = \left(\int_a^b f(x) dx \right) \cdot \left(\int_c^d g(y) dy \right)$

- k) If f is continuous over a region D then $\iint_D f(x, y) dx dy = \iint_D f(r, \theta) \theta dr d\theta$

4. **Surfaces:** Find the domain for the following functions

- a) $f(x, y) = \frac{1}{xy}$ b) $f(x, y) = \frac{1}{x^2 + y^2}$ c) $f(x, y) = \frac{1}{x^2 - y^2}$

5. **Limits and Continuity:** Determine the following limits as $(x, y) \rightarrow (0, 0)$, if they exist.

a) $\lim_{(x,y) \rightarrow (0,0)} \frac{xy+1}{x^2+y^2+1}$

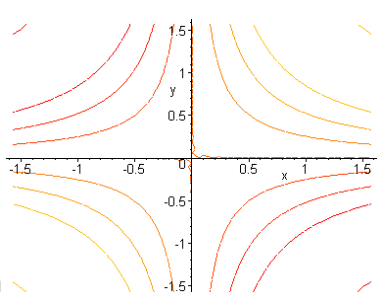
b) $\lim_{(x,y) \rightarrow (0,0)} \frac{xy+1}{x^2+y^2}$

c) $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2+y^2}$

d) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^2+y^2}$

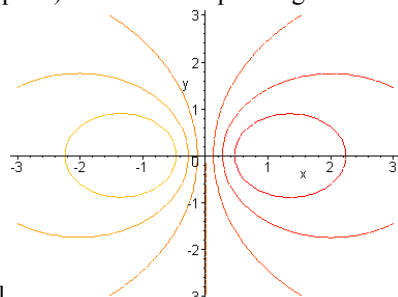
e) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2-y^2}{x^2+y^2}$

6. **Picture:** Match the following contour plots (level plots) to their corresponding surfaces.

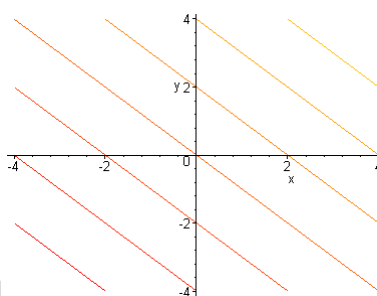


b)

[1]

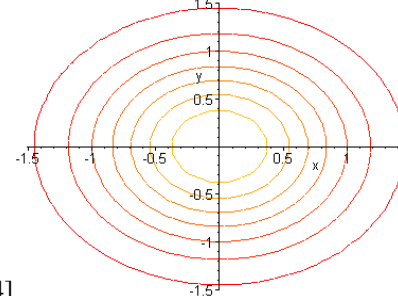


[2]



c)

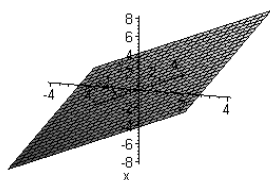
[3]



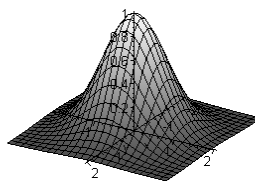
[4]

d)

[A]

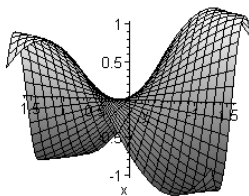


[B]

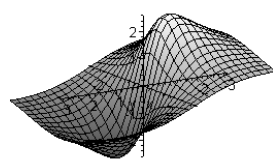


e)

[C]



[D]



Other picture problems:

- Given a contour plot, draw the gradient vector at specific points
- classify some regions as type-1, type-2, or neither.

7. **Differentiation:** Find the indicated derivatives for the given function:

- Find
- Suppose $f(x, y) = 2x^3y^2 + 2y + 4x$, find f_x , f_y , f_{xx} , f_{xy} , f_{yy} , and f_{yx}
- Find the rate of change with respect to y of $x^2 + y^2 + z^2 = 1$ at $P(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$.
- Let $f(x, y) = y^2e^x + y$. Find f_{xyy} , f_{yxy} , and f_{yyx}

8. **Directional Derivatives:**

- Find the directional derivative of $f(x, y) = xy e^{xy}$ at $(-2, 0)$ in the direction of a vector u , where u makes an angle of $\pi/4$ with the x -axis.
- Find $D_u(f)$ where $f(x, y) = \frac{x}{y} - \frac{y}{x}$ and $\vec{u} = \langle -\frac{4}{5}, \frac{3}{5} \rangle$
- Suppose $f(x, y) = x^2e^y$. Find the maximum value of the directional derivative at $(-2, 0)$ and compute a unit vector in that direction.
- For more examples, please see homework assignments

10. **Max/Min Problems:** Compute the extrema as indicated

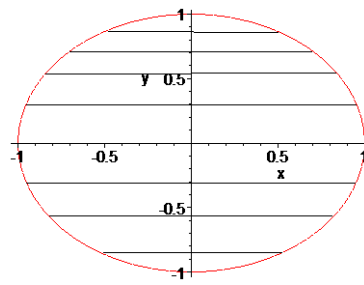
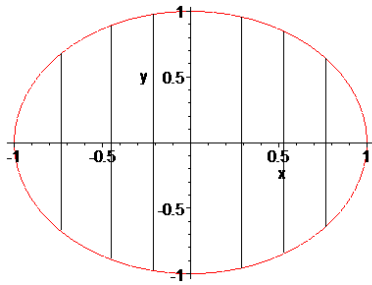
- $f(x, y) = 3x^2 - 2xy + y^2 - 8y$. Find relative extreme and saddle point(s), if any.
- $f(x, y) = 4xy - x^4 - y^4$. Find relative extrema and saddle point(s), if any
- Let $f(x, y) = 3xy - 6x - 3y + 7$. Find absolute maximum and minimum inside the triangular region spanned by the points $(0,0)$, $(3, 0)$, and $(0, 5)$.
- Let $f(x, y) = 3x^2 - 2xy + y^2 - 8y$. Find the absolute extrema over $[0, 1] \times [0, 2]$

11. Evaluate the following integrals:

- $\int_0^1 \int_0^2 xy^2 dx dy$
- $\int_0^\pi \int_0^{\pi/2} \sin(x) \cos(y) dy dx$
- $\int_0^2 \int_{x^2}^x (x^2 + 2y) dy dx$
- $\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2 + y^2} dy dx$
- $\int_0^a \int_0^b \int_0^c x^2 + y^2 + z^2 dx dy dz$
- $\int_0^{\sqrt{\pi}} \int_y^1 \cos(x^2) dx dy$
- $\iint_R \sqrt{x^2 + y^2} dA$, where R is the part of the circle in the 1st quadrant

12. The pictures below show two different ways that a region R in the plane can be covered. Which picture corresponds to the

integral $\iint_R f(x, y) dx dy$



13. Suppose you want to evaluate $\iint_R f(x, y) dA$ where R is the region in the xy plane bounded by $y = 0$, $y = 2 - x^2$, and $y = x$. According to Fubini's theorem you could use either the iterated integral $\iint f(x, y) dx dy$ or $\iint f(x, y) dy dx$ to evaluate the double integral. Which version do you prefer? Explain. You do not need to actually work out the integrals.

14. Use a multiple integral and a convenient coordinate system to find the volume of the solid:

- a) bounded by $z = x^2 - y + 4$, $z = 0$, $y = 0$, $x = 0$, and $x = 4$
- b) bounded by $z = e^{-x^2}$ and the planes $y = 0$, $y = x$, and $x = 1$
- c) bounded above by $z = \sqrt{16 - x^2 - y^2}$ and bounded below by the circle $x^2 + y^2 \leq 4$
- d) evaluate $\iint_R \frac{y}{x^2 + y^2}$ where R is a triangle bounded by $y = x$, $y = 2x$, $x = 2$
- e) bounded by the paraboloid $z = 4 - x^2 - 2y^2$ and the xy plane

15. Find the following surface areas:

- a) of the plane $z = 2 - x - y$ above the rectangle $0 \leq x \leq 2$ and $0 \leq y \leq 3$
- b) of the cylinder $z = 9 - y^2$ above the triangle bounded by $y = x$, $y = -x$, and $y = 3$
- c) of the surface $z = 16 - x^2 - y^2$ above the circle $x^2 + y^2 \leq 9$

16. **Prove** the following facts:

- a) Use the definition to find f_x for $f(x, y) = xy$
- b) Use the definition to find f_x for $f(x, y) = xy$
- c) A function f is said to satisfy the Laplace equation if $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$. Show that the function $f(x, y) = \ln(x^2 + y^2)$ satisfies the Laplace equation.
- d) Two function $u(x, y)$ and $v(x, y)$ are said to satisfy the Cauchy-Riemann equations if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Show that the functions $u(x, y) = e^x \cos(y)$ and $v(x, y) = e^x \sin(y)$ satisfy the Cauchy-Riemann equations.
- e) Let $f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & \text{for } (x, y) \neq (0, 0) \\ 0, & \text{for } (x, y) = (0, 0) \end{cases}$ Then show that f has partial derivatives at $(0, 0)$ but f is not differentiable at $(0, 0)$ – hard!
- f) Prove that the volume of a sphere with radius R is $\frac{4}{3} \cdot \text{Pi} \cdot r^3$
- g) Prove that the surface area of a sphere with radius R is $4 \cdot \text{Pi} \cdot r^2$