Panel 1

$= f(\text{end}) - f(\text{start})$ — line

15. Evaluate $\int_C 2(x+y)dx + 2(x+y)dy$, C curve from (-2, 2) to (4, 3)

Only makes sense if conservative

$F = \langle \overset{M}{2(x+y)}, \overset{N}{2(x+y)} \rangle$

Check: $\frac{\partial M}{\partial y} = \frac{\partial M}{\partial x}$ ✓

Answer: $f(4,3) - f(-2,2)$

Need: potential function $f(x,y) = x^2 + 2xy + y^2$

indep. of path - use line:

$r(t) = \langle -2+6t, 2+1\cdot t \rangle$

$t \in [0,1]$

16. Find the work done by the force field $F = \langle 9x^2y^2, 6x^3y - 1 \rangle$ from P(0,0) to Q(5,9)

$$\int_C F\,dr = \int \langle 9x^2y^2, 6x^3y - 1\rangle \cdot \langle dx, dy \rangle = \int_C 9x^2y^2\,dx + 6x^3y - 1\,dy$$

$$= f(\text{end}) - f(\text{start}) = f(5,9) - f(0,0) = 3\cdot 5^3 \cdot 9^2 - 9$$

$f(x,y) = 3x^3y^2 - y$ is potential function

no const necessary

1

Panel 2

Find potential for $F = \langle \overbrace{2(x+y)}^{f_x}, \overbrace{2(x+y)}^{f_y} \rangle$

$f_x = 2x + 2y \Rightarrow f = x^2 + 2yx + C(y)$

$f_y = 2y + C'(y) = 2x + 2y \Rightarrow C'(y) = 2y \Rightarrow C(y) = y^2 + c$

$\Rightarrow f(x,y) = x^2 + 2xy + y^2 + c$

$F = \langle 9x^2y^2, 6x^3y - 1 \rangle \quad \frac{\partial}{\partial y} M = 18x^2y, \ \frac{\partial}{\partial x} N = 18x^2y$ ✓

$f_x = 9x^2y^2 \Rightarrow f = 3x^3y^2 + C(y)$

$\Rightarrow f_y = \cancel{6x^3}y + C'(y) = \cancel{6x^3}y - 1 \Rightarrow C'(y) = -1, \ C(y) = -y + c$

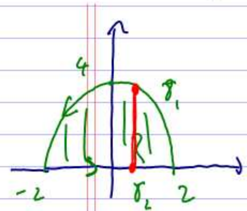
$f(x,y) = 3x^3y^2 - y + c$ is potential function

2

Panel 3

closed curve C

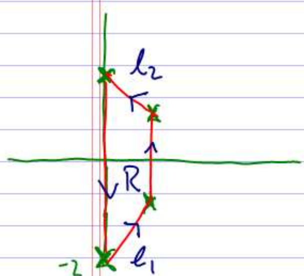
18. Evaluate $\oint_C \overset{M}{2xy}\,dx + \overset{N}{(x+y)}\,dy$ where C bounds the region between $y=0$ and $y=4-x^2$.



Green's thm: $\oint_\gamma M\,dx + N\,dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA$

$= \iint_R 1-2x\,dA = \int_{-2}^{2}\int_{0}^{4-x^2} 1-2x\,dy\,dx = $ HW

21. Evaluate $\oint_C x\sin(y^2) - y^2)dx + (x^2\cos(y^2)+3x)dy$ where C is the boundary of the trapezoid with vertices (0, -2), (1, -1), (1, 1), and (0, 2).



$\oint_C \overbrace{[x\sin y^2 - y^2]}^{M}\,dx + \overbrace{[x^2\cos(y^2)+3x]}^{N}\,dy =$

$\int_0^1\int_{l_1}^{l_2} (2x\cos(y^2)+3) - (2xy\cos(y^2)+2y)\,dy\,dx$

Panel 4

22. Sketch the curve $r(t) = \langle \sin(t), \sin(2t) \rangle$, $t \in [0, 2\pi]$. Evaluate $\oint_\gamma F \cdot dr$ for $F = \langle ye^{x^2}, x^3e^y \rangle$

Panel 5

Recall: Find potential for

a) $\langle \overbrace{6xy^2 - 3x^2}^{M = f_x}, \overbrace{6x^2y + 3y^2 - 7}^{N = f_y} \rangle$

$\frac{\partial}{\partial y} M = 12xy$,

$\frac{\partial}{\partial x} N = 12xy$. ✓

$f_x = 6xy^2 - 3x^2 \Rightarrow f = 3x^2y^2 - x^3 + C(y)$

$\Rightarrow f_y = \boxed{\cancel{6x^2y} + \underline{C'(y)} = \cancel{6x^2y} + 3y^2 - 7}$, $C'(y) = 3y^2 - 7 \Rightarrow C = y^3 - 7y + c$

$\underline{\underline{f(x,y) = 3x^2y^2 - x^3 + y^3 - 7y + c}}$

b) $\langle 2xy, x^2 + z^2, \cancel{10}\,2zy \rangle$

curl $(F) = \vec{0}$: $\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2+z^2 & \cancel{10}\,2zy \end{vmatrix} = \langle \cancel{8z}\,0, 0, 2x - 2x \rangle$ ✓

5

Panel 6

Find $\int_C xy\,dx + x^3 - 2y\,dy$, C line from $(1,1)$ to $(3,4)$

Do it: $\gamma(t) = \langle \underbrace{1 + 2\cdot t}_{x}, \underbrace{1 + 3\cdot t}_{y} \rangle$, $t \in [0,1]$

$\Rightarrow \int_C (xy)\,dx + (x^3 - 2y)\,dy =$

$= \int_0^1 (1+2t)(1+3t)\,2dt + \left[(1+2t)^3 - 2(1+3t)\right] 3dt =$

$= \int_0^1 2(1+2t)(1+3t) + 3\left((1+2t)^3 - 2(1+3t)\right) dt =$ <u>done</u>

<u>basically!</u>

6

Panel 7

Quiz 9 - Part 1

1. Which of the following vector fields are conservative? If it is, find its potential function.

a) $F(x, y) = < y^2 + y, 2xy + x >$

b) $F(x, y, z) = < xz, 1 - 6yz^2, x^2 - 9y^2z^z >$

back at 9:30

7

Panel 8

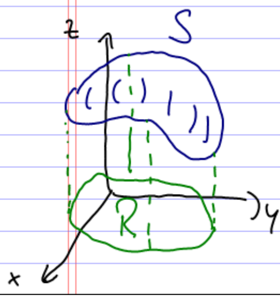
Surface Integrals $\quad r(t) = \langle x(t), y(t) \rangle \quad \gamma: y = f(x)$

Know: $\int_\gamma ds = \int \sqrt{(x')^2 + (y')^2}\, dt = \int \sqrt{(f')^2 + 1}\, dx$ (length)

$\Gamma: y = f(x)$

Also: $\int_\gamma g(x,y)\, ds = \int g(x(t), y(t)) \sqrt{x'^2 + y'^2}\, dt = \int_a^b g(x, f(x)) \sqrt{f'^2 + 1}\, dx$

Now: $\iint_S g(x,y,z)\, dS$ is a surface integral of $w = g(x,y,z)$ over surface $S: z = f(x,y)$



$$\iint_S g(x,y,z)\, dS = \iint_R g(x, y, f(x,y)) \sqrt{f_x^2 + f_y^2 + 1}\, dA$$

$$\left[\iint dS = \iint \sqrt{f_x^2 + f_y^2 + 1}\, dA \right]$$

surface area

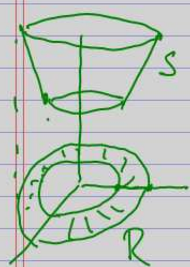
8

Panel 9

Ex: Evaluate $\iint_S x^2 z \, dS$, S portion of $z^2 = x^2 + y^2$ between $z = 1$ and $z = 4$.

$z = \sqrt{x^2+y^2} = f(x,y)$

$$\iint_S x^2 z \, dS = \iint_R x^2 \underbrace{f(x,y)} \underbrace{\left(\sqrt{f_x^2 + f_y^2 + 1}\right) dA} =$$



$$\left(f_x = \frac{x}{\sqrt{x^2+y^2}}, \quad f_y = \frac{y}{\sqrt{x^2+y^2}} \right.$$

$$\left. \Rightarrow f_x^2 + f_y^2 + 1 = \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1 = 2 \right)$$

$$= \iint_R x^2 \sqrt{x^2+y^2} \sqrt{2} \, dA =$$

$x = r\cos\theta$

$y = r\sin\theta$

$dA = r\,dr\,d\theta$

$$= \int_0^{2\pi} \int_1^2 r^2 \cos^2\theta \, r \sqrt{2} \, r \, dr \, d\theta = \frac{1023}{5} \sqrt{2} \pi$$

9

Panel 10

Surface integrals have functions as integrand. Want to extend that integral to vector fields:

Def: If S is surface given by $z = f(x,y)$, over region R.

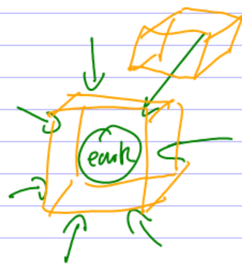
$\Rightarrow \iint_S \vec{F} \cdot \vec{n} \, dS$ is the flux of a vector field $\vec{F}$

where $\vec{n}$ is the normal vector to the surface S, given by $\vec{n} = \frac{1}{\sqrt{f_x^2 + f_y^2 + 1}} \langle -f_x, -f_y, 1 \rangle$

Note: Think of S (surface) as being submerged in vector field F (fluid). Then the flux integral measures how much liquid flows through S per time unit.

10

Panel 11



11

Panel 12

Ex: Let S be $z = 9 - x^2 - y^2$, $z \geq 0$ and $\vec{F} = \langle 3x, 3y, z \rangle$.
Find flux of $\vec{F}$ through S.

$$\iint_S \vec{F} \cdot \vec{n} \, dS \qquad \vec{n} = \frac{1}{\sqrt{f_x^2 + f_y^2 + 1}} \langle -f_x, -f_y, 1 \rangle, \quad f(x,y) = 9 - x^2 - y^2$$

$$\Rightarrow f_x = -2x, \; f_y = -2y \Rightarrow \vec{n} = \frac{1}{\sqrt{4x^2 + 4y^2 + 1}} \langle +2x, 2y, 1 \rangle$$

$$dS = \sqrt{f_x^2 + f_y^2 + 1} \, dA = \sqrt{4x^2 + 4y^2 + 1} \, dA$$

$$\Rightarrow \iint_S \vec{F} \cdot \vec{n} \, dS = \iint_R \langle 3x, 3y, z \rangle \cdot \frac{1}{\cancel{\sqrt{}}} \langle 2x, 2y, 1 \rangle \cdot \cancel{\sqrt{}} \, dA =$$

$$= \iint_R 6x^2 + 6y^2 + z \, dA$$

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Panel 13

What is R if $\underline{z = 9 - x^2 - y^2}$, $z \ge 0$

Thus:

$$\iint_R 6x^2 + 6y^2 + z \, dA = \iint_R 6x^2 + 6y^2 + 9 - x^2 - y^2 \, dA$$

$$\int_0^{2\pi}\int_0^3 5x^2 + 5y^2 + 9 \qquad r\,dr\,d\theta =$$

$x = r\cos\theta$

$y = r\sin\theta$

$$\int_0^{2\pi}\int_0^3 (5r^2 + 9)\, r\,dr\,d\theta = \frac{567}{2}\pi$$

Short cut

13

Panel 14

$$\int_C ds = \int_a^b \sqrt{(x')^2 + (y')^2}\,dt \quad \leftarrow C: \langle x(t), y(t) \rangle$$

$$\int_C ds = \int_a^b \sqrt{[f'(x)]^2 + 1}\,dx \quad \leftarrow C: y = f(x)$$

$$\iint_R dS = \sqrt{f_x^2 + f_y^2 + 1}\,dA \qquad \begin{pmatrix}\text{surface}\\ \text{area}\end{pmatrix}$$

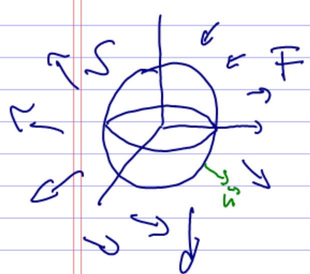
14

Panel 15

The Divergence Thm (Gauss Thm)

Let Q be a region in $\mathbb{R}^3$ bounded by a closed surface S, with outer normal vector $\vec{n}$. If $\vec{F}$ is a vector field with continuous derivatives, then:

$$\oiint_S \vec{F}\cdot\vec{n}\, dS = \iiint_Q \underbrace{\nabla\cdot\vec{F}}_{\operatorname{div}(F)}\, dV$$



Greens

$\langle M, N, 0\rangle$

$$\oint \vec{F}\, d\vec{r} = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dA$$

$$= \iint_R \operatorname{curl}(\vec{F})\, dA$$

15

Panel 16

Ex: Let Q be bdd by $x^2+y^2=4$, $z\in[0,3]$. Let S be the surface of Q and define $\vec{F}=\langle x^3, y^3, z^3\rangle$. Find

$$\iint_S \vec{F}\cdot\vec{n}\, dS$$

16

Panel 17

Ex: Q region bdd by $z = 4 - x^2$, $y + z = 5$, xy and xz-planes.

Let $\vec{F} = \langle x^3 + \sin(z), x^2 y + \cos(z), e^{x^2+y^2} \rangle$, find $\iint_R \vec{F} \cdot \vec{n}\, dS$

17

Panel 18

Stoke's Theorem

S a surface with unit normal $\vec{n}$ bdd by piecewise smooth curve C with positive orientation. If F is a vector field with cont. diff'ble partials then

$$\int \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl}(F) \cdot \vec{n}\, dS$$

Ex: C is triangle bdd by coord. plane and $2x + 2y + z = 6$. Let $F = \langle -y^2, z, x \rangle$ and find $\int_C \vec{F} d\vec{r}$.

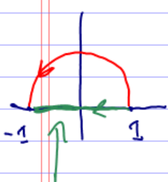
18

Panel 19

From practice 3 $\int F\,dr = f(-1,0) - f(1,0) = -1-1 = \underline{-2}$

$\int F\,dr$, $F = \langle y^3+1, 3xy^2+1 \rangle$

check: $\frac{\partial M}{\partial y} = 3y^2 \stackrel{?}{=} \frac{\partial N}{\partial x} = 3y^2$ ✓ conserv.



Thus: $\int F\,dr = f(B) - f(A)$

$\gamma(t) = \langle t, 0 \rangle$, t from 1 to -1

$$\Rightarrow \int_\gamma (y^3+1)\,dx + (3xy^2+1)\,dy = \int_1^{-1} 1\,dt = -2$$

$f_x = y^3+1$

$\Rightarrow f = xy^3 + x + C'(y) \rightarrow f_y = 3xy^2 + C'(y) = 3xy^2+1 \Rightarrow C'(y) = 1 \quad C(y) = y + c$

$f(x,y) = xy^3 + x + y + c$

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