Panel 1

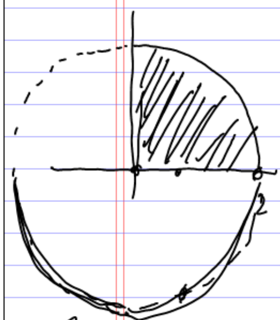
$$\int_0^2 \int_0^{\sqrt{4-x^2}} e^{x^2+y^2}\,dy\,dx = \int_0^{\pi/2}\int_0^2 r e^{r^2}\,dr\,d\theta =$$

$x = r\cos\theta$

$y = r\sin\theta$

$dy\,dx = r\,dr\,d\theta$

$$= \int_0^{\pi/2} \frac{1}{2} e^{r^2}\Big|_{r=0}^{2} = \int_0^{\pi/2} \frac{1}{2}\left(e^4 - 1\right) d\theta = \frac{\pi}{4}\left(e^4-1\right)$$



$-\sqrt{4-x^2}$

$y = \sqrt{4-x^2}$

$y^2 = 4 - x^2$

$x^2 + y^2 = 4$

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\,dx \quad \text{(full circle)}$$

1

Panel 2

Last time

Center of Mass:

$M_x = \iint_R y\,\rho(x,y)\,dA$ $\qquad \bar{x} = M_y / M$

$M_y = \iint_R x\,\rho(x,y)\,dA$ $\qquad \bar{y} = M_x / M$

$M = \iint_R \rho(x,y)\,dA$

HW find surface of ball

Surface Area: $\iint dS = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\,dA$

$z = f(x,y)$

Vector Field: $\vec{F}(x,y,z) = \langle M, N, P\rangle$, $M = M(x,y,z)$, $N = N(x,y,z)$, $P = P(x,y,z)$

curl $(\vec{F})$: $\vec{\nabla} \times \vec{F}$

div $(\vec{F})$: $\vec{\nabla} \cdot \vec{F}$

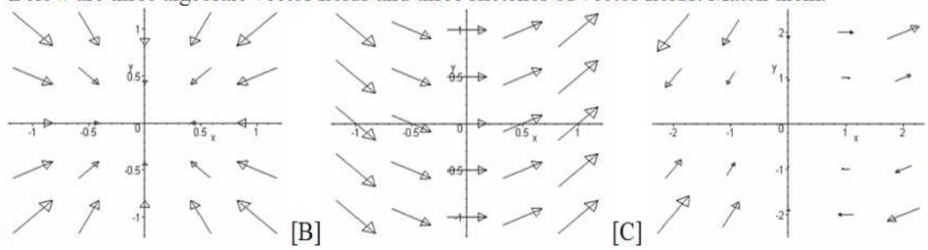
conservative: if $\vec{F} = \nabla f$, $f(x,y,z)$ function

2

Panel 3

Quiz 8 - Part 1

1. Below are three algebraic vector fields and three sketches of vector fields. Match them.



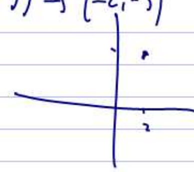
[A] [B] [C]

$(2,3) \to$

(1) $F(x,y) = \langle xy, y(x-1) \rangle$

(2) $F(x,y) = \langle 1, x \rangle$ $\quad (2,3) \to (1,2)$

(3) $F(x,y) = \langle -x, -y \rangle$ $\quad (2,3) \to (-2,-3)$



3

Panel 4

Quiz 8 - Part 2

2. Suppose that $F(x,y,z) = \langle x^3 z, x^2 z, xy \rangle$ is some vector field.

a) Find $div(F)$ $= \vec{\nabla} \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle x^3 z, x^2 z, xy \rangle =$

$$= \frac{\partial}{\partial x}(x^3 z) + \frac{\partial}{\partial y}(x^2 z) + \frac{\partial}{\partial z}(xy)$$

b) Find $curl(F)$ $= \vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 z & x^2 z & xy \end{vmatrix} =$

$$= \left\langle \frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(x^2 z), -\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(x^3 z)\right], \frac{\partial}{\partial x}(x^2 z) - \frac{\partial}{\partial y}(x^3 z) \right\rangle$$

4

Panel 5

Quiz 8 - Part 3

3. Find the conservative vector field with $f(x,y,z) = ze^y \sin(xz)$ as potential function.

$$\nabla f = \vec{F}$$

$$\left\langle \frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f, \frac{\partial}{\partial z} f \right\rangle = \langle z^2 e^y \cos(xz), \quad , \quad \rangle$$

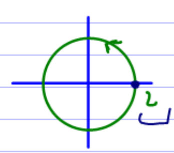
Next up: Line integrals

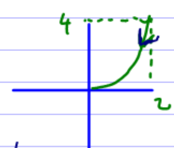
$$\int_C \underbrace{?}\, d\underbrace{?} \quad , C \text{ curve}$$

5

Panel 6

Before we continue, need to find paths: Find $r(t) = \langle x(t), y(t) \rangle$ expressions for these paths:





$r(t) = \langle 2\cos(t), 2\sin(t) \rangle$
$t = 0$ to 2π

$r(t) = \langle t, t^2 \rangle$
t from 2 to 0

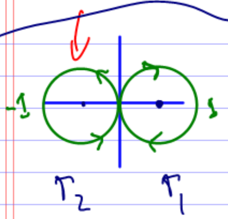
$r_1(t) = \langle t, -t+1 \rangle$, $t = 1$ to 0 $(y = -x+1)$

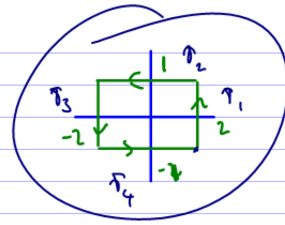
$r_2(t) = \langle t, t+1 \rangle$, $t = 0$ to -1 $(y = x+1)$

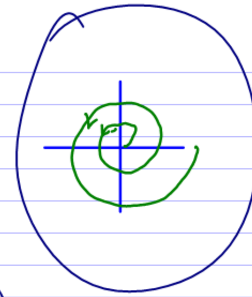
$r_3(t) = \langle t, 0 \rangle$, $t = -1$ to 1

6

Panel 7







$r_1(t) = \langle \frac{1}{2} + \frac{1}{2}\cos(t), \frac{1}{2}\sin(t) \rangle$, $t = \pi$ to $-\pi$

$r_2(t) = \langle -\frac{1}{2} + \frac{1}{2}\cos(t), \frac{1}{2}\sin(t) \rangle$, $t = 0$ to 2π

$\langle t\cos(t), t\sin(t) \rangle$

Square: $r_1(t) = \langle 2, t \rangle$, $t = -1, 1$ | $r_3(t) = \langle -2, t \rangle$, $t = 1$ to -1

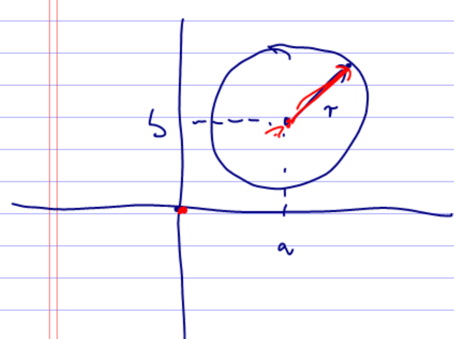
$r_2(t) = \langle t, 1 \rangle$, $t = 2, -2$ | $r_4(t) = \langle t, -2 \rangle$, $t = -2$ to 2

7

Panel 8

$\langle r\cos(t), r\sin(t) \rangle$ $t = 0$ to 2π

circle, radius r, once around



$\langle a, b \rangle + r\langle \cos(t), \sin(t) \rangle$

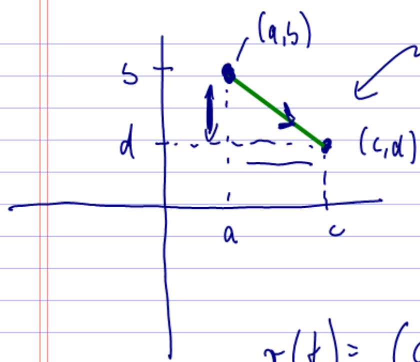
$\langle a + r\cos(t), b + r\sin(t) \rangle$

$t = 0$ to 2π

[or $(x-a)^2 + (y-b)^2 = r^2$]

8

Panel 9



$y = \left(\frac{d-b}{c-a}\right)\cdot x + (\text{y-intercept})$

as vector: $\langle c-a, d-b\rangle \cdot t$

$r(t) = (a,b) + t\langle c-a, d-b\rangle$, $t = 0$ to 1

9

Panel 10

3 Common Parametrizations

a) Circle, radius r, center (a,b):

$$r(t) = \langle a + r\cos(t), b + r\sin(t)\rangle, \quad t \text{ from } 0 \text{ to } 2\pi$$

b) Line segment from (a,b) to (c,d):

$$r(t) = \langle a,b\rangle + t\langle c-a, d-b\rangle =$$
$$= \langle a + t(c-a), b + t(d-b)\rangle, \quad t \text{ from } 0 \text{ to } 1$$

c) function $y = f(x)$

$$r(t) = \langle t, f(t)\rangle$$

10

Panel 11

Def: C a curve $r(t) = \langle x(t), y(t) \rangle$, $t \in [a,b]$

$$\Rightarrow \int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\,\underline{\sqrt{(x')^2+(y')^2}}\,dt$$

(line integral of f along curve C)

$$\Rightarrow \int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

(line integral of f along curve C with respect to x)

$$\Rightarrow \int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

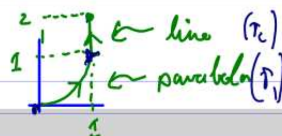
(line integral of f along curve C with respect to y)

$$\Rightarrow \int_C f(x,y)\,dx + g(x,y)\,dy = \int_C f(x,y)\,dx + \int_C g(x,y)\,dy$$

11

Panel 12

Ex: Evaluate $\int_C 2x\,ds$ where C:



$r_1(t) = \langle t, t^2 \rangle$, $t = 0$ to 1

line through $(1,1)$ to $(1,2)$: $r(t) = \langle 1, 1+t \rangle$, $t = 0$ to 1

$$\int_C 2x\,ds = \int_{r_1} 2x\,ds + \int_{r_2} 2x\,ds =$$

$$= \int_0^1 2t\sqrt{\underbrace{1}_{(x')^2} + \underbrace{4t^2}_{(y')^2}}\,dt + \int_0^1 2\cdot 1\sqrt{\underbrace{0^2}_{(x')^2} + \underbrace{1^2}_{(y')^2}}\,dt$$

easy!

12

Panel 13

Evaluate $\int_C x y^2 \, ds$, where C is circle, radius 3.

$r(t) = \langle 3\cos(t), 3\sin(t) \rangle$, $t = 0$ to 2π

$$\int_C \underline{x y^2} \, ds = \int_0^{2\pi} \underbrace{3\cos(t)}_{x} \cdot \underbrace{9\sin^2(t)}_{y^2} \sqrt{\underbrace{9\sin^2(t)}_{(x')^2} + \underbrace{9\cos^2(t)}_{(y')^2}} \, dt =$$

$$= 27 \int_0^{2\pi} \cos(t)\sin^2(t) \sqrt{9} \, dt =$$

$$= 81 \int_0^{2\pi} \underline{\cos(t)} \, \underbrace{\sin^2(t)}_{u^2} \, \underbrace{dt}_{du} = 81 \cdot \frac{1}{3} \sin^3(t) \Big|_0^{2\pi} = \underline{\underline{0}}$$

$u = \sin(t)$

$du = \cos(t) dt$

13

Panel 14

Def: C a curve $r(t) = \langle x(t), y(t) \rangle$, $t \in [a,b]$

$\Rightarrow \int_C f(x,y) \, ds = \int_a^b f(x(t), y(t)) \sqrt{(x')^2 + (y')^2} \, dt$

(line integral of f along curve C)

$\Rightarrow \int_C f(x,y) \, dx = \int_a^b f(x(t), y(t)) \frac{dx}{dt} \, dt$

(line integral of f along curve C with respect to x)

$\Rightarrow \int_C f(x,y) \, dy = \int_a^b f(x(t), y(t)) \frac{dy}{dt} \, dt$

(line integral of f along curve C with respect to y)

$\Rightarrow \int_C f(x,y) \, dx + g(x,y) \, dy = \int_C f(x,y) \, dx + \int_C g(x,y) \, dy$

14

Panel 15

Ex: Find $\boxed{\int_C y^2\,dx + x\,dy}$ where (a) C_1 is line from $(-5,-3)$ to $(0,2)$ and (b) C_2: $x = 4-y^2$ from $(-5,-3)$ to $(0,2)$

(a) $\int_{C_1} \boxed{y^2\,dx} + \boxed{x\,dy}$, from $(-5,-3)$ to $(0,2)$

$r(t) = \langle -5+5t, -3+5t\rangle$, $t = 0$ to 1

$$\int_0^1 \underbrace{(-3+5t)^2}_{y^2} \underbrace{\underbrace{5}_{x'}\,dt}_{dx} + \int_0^1 \underbrace{(-5+5t)}_{x} \underbrace{\underbrace{5}_{y'}\,dt}_{dy} = \int_0^1 (-3+5t)^2\cdot 5\,dt + \int_0^1 (-5+5t)\,5\,dt$$

= Answer in Maple

$ds = \cancel{\sqrt{(x')^2 + \cancel{(y')^2}}}\,dt$

dx (no difference in y)

15

Panel 16

Ex: Find $\boxed{\int_C y^2\,dx + x\,dy}$ where (a) C_1 is line from $(-5,-3)$ to $(0,2)$ and (b) C_2: $x = 4-y^2$ from $(-5,-3)$ to $(0,2)$

from $(-5,-3)$ to $(0,2)$

(b) $r_2(t) = \langle 4-t^2, t\rangle$ $t = -3$ to 2 ✓

$dx = \frac{dx}{dt}\,dt$

$$\int_C y^2\,dx + x\,dy =$$

$$\int_{-3}^{2} t^2(-2t)\,dt + (4-t^2)\,1\cdot dt$$

16

Panel 17

Ex: Find $\int_c y^2\, ds$ where $C_2:\ x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$

from $(-5,-3)$ to $(0,2)$

(5) $r_2(t) = \langle 4-t^2, t \rangle$ $t = -3$ to 2 ✓

$dx = \frac{dx}{dt}\, dt$

$$\int_C y^2\, ds =$$

$$\int_{-3}^{2} t^2 \sqrt{4t^2+1}\, dt$$

17

Panel 18

Line Integrals of Vector Fields:

Suppose F is a vector field on a smooth curve C, defined via $\vec{r}(t)$, $a \le t \le b$. Then the line integral of F along C is:

dot product

$$\int_C \vec{F} \cdot d\vec{r}$$

$$\int_C \langle M, N \rangle \cdot \langle dx, dy \rangle = \int_C M\,dx + N\,dy$$

$$\int_C \langle M, N, P \rangle \cdot \langle dx, dy, dz \rangle = \int_C M\,dx + N\,dy + P\,dz$$

18

Panel 19

Ex: Let $\vec{F}(x,y) = \langle x^2, -xy \rangle$ C quarter circle radius 1.

Find $\int_C \vec{F} \cdot d\vec{r}$

C is given by $r(t) = \langle \cos(t), \sin(t) \rangle$, $t = 0$ to $\pi/2$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \langle x^2, -xy \rangle \cdot \langle dx, dy \rangle =$$

$$= \int_C x^2\,dx - xy\,dy =$$

$$= \int_0^{\pi/2} (\cos(t))^2 (-\sin(t))\,\underline{dt} - \cos(t)\cdot\sin(t)\cos(t)\,\underline{dt} =$$

$$= \int_0^{\pi/2} -\sin(t)\cos^2(t) - \sin(t)\cos(t)\,dt = \text{Easy}$$

19

Panel 20

Note: $\int_C \vec{F} \cdot d\vec{r}$ is the work done to move a particle along path C through a force field $\vec{F}$.

Ex: Find work required to move particle from (2,0,0) to (3,4,5) through force field $\vec{F} = \langle y, z, x \rangle$

$$r(t) = \langle 2 + \underbrace{1}\,t,\ 0 + \underbrace{4}\,t,\ 0 + \underbrace{5}\,t \rangle,\ t = 0 \text{ to } 1$$

$$= \langle 2+t, 4t, 5t \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \int \langle y, z, x \rangle \langle dx, dy, dz \rangle = \int y\,dx + z\,dy + x\,dz =$$

$$= \int_0^1 4t\,dt + 5t\cdot 4\,dt + (2+t)\cdot 5\,dt$$

20

Panel 21

Need short cut for integration: Suppose $\vec{F}$ is a conservative vector field, C a path from A to B where $r(a)=A$ and $r(b)=B$. Find short cut for

$$\int_C \vec{F}\cdot d\vec{r} = (\text{antideriv. of } F)(\text{point } B) - (\text{antideriv})(\text{point } A)$$

If $\vec{F}$ is conservative then there is function f with $\nabla f = \vec{F}$. That f is the potential function and is similar to antideriv:

$$\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A)$$

21

Panel 22

Ex. $\vec{F}(x,y) = \langle x,y\rangle$, $\boxed{f(x,y)=\frac{1}{2}(x^2+y^2)}$, C line from $(1,1)$ to $(3,4)$. Find $\int_C \vec{F}\cdot d\vec{r}$
$\qquad A \qquad B$

old way: $r(t) = \langle 1+2t, 1+3t\rangle$, $t=0$ to 1

$$\Rightarrow \int_C \vec{F}\, d\vec{r} = \int_C x\,dx + y\,dy = \int_0^1 (1+2t)\cdot 2\,dt + (1+3t)3\,dt =$$

$$= \int_0^1 2+4t+3+9t\,dt = \int_0^1 5+13t\,dt = 5+\frac{13}{2} = \underline{\underline{\frac{23}{2}}}$$

new way: is f potential? $\nabla f = \langle x, y\rangle = \vec{F}$

$$\Rightarrow \int_C \vec{F}\, d\vec{r} = f(B) - f(A) =$$

$$= \frac{1}{2}(9+16) - \frac{1}{2}2 = \underline{\underline{\frac{23}{2}}}$$

22

Panel 23

Find potential function for $F(x,y) = (x^2+y^2, 2xy)$

Is it conservative? Yes if it had a potential f.

$\Rightarrow \nabla f = F \iff \frac{\partial}{\partial x} f = x^2+y^2$ and $\frac{\partial}{\partial y} f = 2xy$

I need f s.t. $\frac{\partial}{\partial x} f = x^2+y^2$

$$\Rightarrow f(x,y) = \int x^2+y^2 \, dx = \frac{1}{3}x^3 + xy^2 + C(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = 2xy + C'(y) = 2xy$$

$\boxed{f(x,y) = \frac{1}{3}x^3 + xy^2 + c \text{ is potential!}}$

$$\Rightarrow C'(y) = 0 \Rightarrow C(y) = \text{const.}$$

23

Panel 24

Find potential function for $\vec{F} = (3+2xy, x^2-3y^2)$ if exists

Looking for $f(x,y)$ s.t. $\frac{\partial}{\partial x} f = 3+2xy$

$$\Rightarrow f(x,y) = 3x + x^2y + C(y) \quad (1)$$

and s.t. $\frac{\partial}{\partial y} f = x^2 - 3y^2$. But

$$\frac{\partial}{\partial y} f = x^2 + C'(y) \overset{\text{supposed}}{=} x^2 - 3y^2$$

$$\Rightarrow C'(y) = -3y^2 \Rightarrow C(y) = -y^3 + c$$

$$\Rightarrow f(x,y) = 3x + x^2y - y^3 + c$$

$\frac{\partial}{\partial x} f = 3+2xy$

$\frac{\partial}{\partial y} f = x^2 - 3y^2$

24

Panel 25

Find potential function for $\langle x^2\cos(y), -y^2\sin(x)\rangle$

Quiz:

$\int_C f \, dr$

$\int_C f \, dx$ $\qquad \int_C f\,dx + g\,dy$

$\int_C f \, dy$

$\int_C \vec{F} \, d\vec{r}$

and conserv. F with potent. f

25