Panel 1

Partial Deriv. Review

1a) $f(x,y) = xy$

$f_x = y \quad , \quad f_y = x \quad , \quad \nabla f = \langle f_x , f_y \rangle = \langle y, x \rangle$

b) $f(x,y) = x/y = \boxed{x}\, y^{-1}$

$f_x = 1/y$ $\qquad (x/5 = \frac{1}{5} x)$

$f_y = -x y^{-2}$ $\qquad (5 y^{-1} \quad)$

c) $f(x,y,z) = \dfrac{x^2}{y^2+z^2} = x^2 (y^2+z^2)^{-1}$

$f_x = \dfrac{2x}{y^2+z^2}$ $\qquad (\frac{x^2}{17}$

$f_y = x^2 (-1) (y^2+z^2)^{-2} \cdot 2y$

$f_z = x^2 (-1) (y^2+z^2)^{-2} \cdot 2z$

1

Panel 2

d) $f(x,y,z) = \dfrac{xy}{yz+z^2} = x \dfrac{y}{yz+z^2} = xy \, \underline{(yz+z^2)^{-1}}$

$f_x = \dfrac{y}{yz+z^2}$

$f_y = \dfrac{x(yz+z^2) - xyz}{(yz+z^2)^2}$

$f_z = xy (-1) (yz+z^2)^{-2} \cdot (y+2z)$

e) $f(x,y) = x \sin(y)$

$f_x = \sin(y)$

$f_y = x \cos(y)$

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Panel 3

f) $f(x,y) = xy \sin(xy)$

$f_x = y \cdot \sin(xy) + xy \cdot \cos(xy) \cdot y$

$f_y = x \sin(xy) + xy \cdot \cos(xy) x$

g) $f(x,y,z) = x \cdot \cos\left(\frac{x}{z}\right) = x \cos(x z^{-1})$

$f_x = \cos(x/z) + x\left(-\sin\left(\frac{x}{z}\right)\right)\frac{1}{z}$

$f_y = 0$

$f_z = x\left(-\sin(xz^{-1})\right) \cdot x(-1)z^{-2}$

3

Panel 4

h) $f(x,y,t) = xy\sqrt{1+t^2} = xy\,(1+t^2)^{1/2}$

$f_x = y\,(1+t^2)^{1/2}$

$f_y = x\,(1+t^2)^{1/2}$

$f_t = xy\,\frac{1}{2}\left(1+t^2\right)^{-1/2} \cdot (2t)$

i) $f(x,y,t) = xy\sqrt{x^2+t^2}$

$f_x = y\sqrt{x^2+t^2} + xy\,\frac{1}{2}\left(x^2+t^2\right)^{-1/2}(2x)$

$f_y = x\sqrt{x^2+t^2}$ $\qquad$ $f_t = xy\,\frac{1}{2}\left(x^2+t^2\right)^{-1/2}(2t)$

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Panel 5

② a) $f(x,y) = e^{-(x^2+y^2)}$ Find f_{xx}

$$f_x = \underbrace{-2x}\cdot \underbrace{e^{-(x^2+y^2)}}$$

$$f_{xx} = -2\cdot e^{-(x^2+y^2)} - 2x\, e^{-(x^2+y^2)}(-2x)$$

b) $f(x,y) = x \ln(y+x)$, find f_{yy}

$$f_y = x \frac{1}{(y+x)} \cdot 1 = x(y+x)^{-1}$$

$$f_{yy} = x(-1)(y+x)^{-2}\cdot(1)$$

$$\frac{d}{dx}\ln(x) = \frac{1}{(x)}\cdot 1$$

$$f(x,y) = x\ln(x-y)$$

$$f_y = x \cdot \frac{1}{x-y}\cdot(-1)$$

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Panel 6

c) $f(x,y) = x^4 + 3x^2y^3 + 2y^5$, $x = \cos(t)$, $y = \sin(t)$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$\longrightarrow (4x^3 + 6xy^3)\cdot(-\sin(t)) + (9x^2y^2 + 10y^4)\cdot\cos(t)$$

$$= (4\cos^3(t) + 6\cos(t)\sin^3(t))(-\sin(t)) + \text{blah} \ldots$$

$$\frac{\partial f}{\partial t} \text{ when } t = \pi : (4x^3 + 6xy^3)(-\sin(t)) + (9x^2y^2 + 10y^4)\cos(t)\Big|_{\substack{t=\pi\\ x=-1\\ y=0}}$$

$$= (-4)\,0 + 0\cdot(-1) = \underline{0}$$

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Panel 7

d) $f(x,y) = x^2y^3 + \frac{x}{y} \sim xy^{-1}$, $x = t^2+s^2$, $y = ts$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} =$$

$$= \left(2xy^3 + \frac{1}{y}\right)(2t) + \left(3x^2y^2 - xy^{-2}\right)(s)$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} =$$

$$= \left(2xy^3 + \frac{1}{y}\right)(2s) + \left(3x^2y^2 - xy^{-2}\right)t.$$

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Panel 8

Review

Chain Rule:

① $z = F(x,y)$, $x = x(t)$, $y = y(t)$

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial t}$$

② $z = F(x,y)$, $x = x(s,t)$, $y = y(s,t)$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

Directional Derivative of $F(x,y)$ in direction of unit vector $\vec{u}$

$$\left(\lim_{h \to 0} \frac{F(x+hu_1, y+hu_2) - F(x,y)}{h} = \right) D_{\vec{u}} f = \nabla F \cdot \vec{u} = \langle f_x , f_y \rangle \cdot \vec{u}$$

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Panel 9

The Gradient of $z = f(x,y)$

$$\nabla f = \langle f_x, f_y \rangle$$

Properties of Gradient

- The gradient is a vector!
- Gradient is perp. to level curves
- Gradient points in direction of steepest increase
- $\|\nabla f\|$ is the steepest increase

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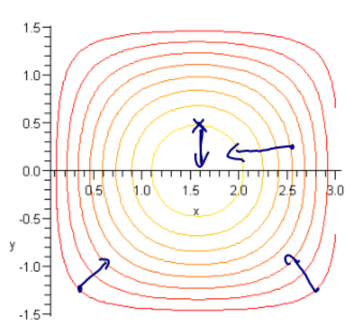
Panel 10

Useful Theorem: The max. value of $|D_{\vec{u}} f|$ is achieved if $\vec{u}$ points in the direction of ∇f. The maximum value is $\|\nabla f\|$.



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Panel 11

$z = x^2 y + xy^2$, $x = 2+t^4$, $y = 1-t^3$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} =$$

$$= \left(2xy + y^2\right) 4t^3 + \left(x^2 + 2xy\right)\left(-3t^2\right)$$

Suppose $\sqrt{xy} = 1 + x^2 y$ defines y as $y = y(x)$. Find $\frac{\partial y}{\partial x}$

$\Rightarrow F(x,y) = 1 + x^2 y - \sqrt{xy} = 0$, $x = x$, $y = y(x)$

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} = 0 \qquad \Rightarrow \frac{dy}{dx} = -\frac{2xy - \frac{1}{2}(xy)^{-1/2} y}{x^2 - \frac{1}{2}(xy)^{-1/2} x}$$

$$\left(2xy - \frac{1}{2}(xy)^{-1/2} \cdot y\right) \cdot 1 + \left(x^2 - \frac{1}{2}(xy)^{-1/2} \cdot x\right)\left(\frac{\partial y}{\partial x}\right) = 0$$

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Panel 12

Rate of change of $f(x,y) = y \ln(x)$ at $P(1,-3)$ in dir $\langle -\frac{4}{5}, \frac{3}{5} \rangle$

Deriv of f in $\langle -\frac{4}{5}, \frac{3}{5} \rangle$ at $P(1,-3)$

$$= \left\langle \frac{y}{x}, \ln(x) \right\rangle\Big|_{(1,-3)} \cdot \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle = \langle -3, 0 \rangle \cdot \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle = \frac{12}{5}$$

Find ∇f if $f(x,y,z) = \ln(xy^2 z^3)$ and total differential

Gradient: $\langle f_x, f_y, f_z \rangle = \left\langle \frac{1}{xy^2z^3} \cdot y^2 z^3, \frac{1}{xy^2z^3} \cdot 2yxz^3, \frac{1}{xy^2z^3} \cdot 3z^2 xy^2 \right\rangle$

$$= \frac{z^2 y}{xy^2 z^3} \langle yz, 2xz, 3xy \rangle$$

Total diff: $dF = f_x\,dx + f_y\,dy + f_z\,dz = \frac{z^2 y}{xy^2 z^3}\left(yz\,dx + 2xz\,dy + 3xy\,dz\right)$

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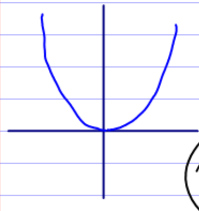
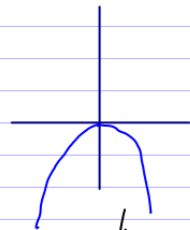
Panel 13

Review of Max/Min Problem in $\mathbb{R}$



min

$y = x^2$



max

$y = -x^2$

$x=0$

neither

$y = x^3$

$f'(x) = 2x$

$f'(x) = 0 \Rightarrow x = 0$

$f''(x) = 2$

① find $f'(x)$

② solve $f'(x) = 0$ are critical

③ Decide on max/min

if $f'' > 0 \Rightarrow$ min

if $f'' < 0 \Rightarrow$ max

if f'' = 0 $\Rightarrow$ "No Clue!"

$f'(x) = 3x^2$

$f''(x) = 0 \Rightarrow x = 0$

1st deriv test also useful

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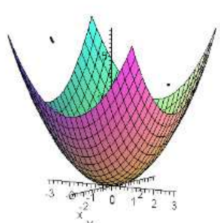
Panel 14

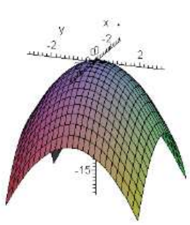
Life in 3D

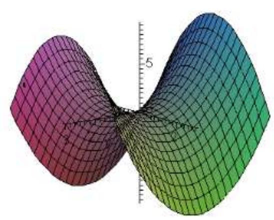
$f(x,y) = x^2 + y^2$ (min)

$f(x,y) = -x^2 - y^2$ (max)

$f(x,y) = x^2 - y^2$ (saddle)







① Take $\nabla f = (f_x, f_y)$

② Solve $\nabla f = 0 \leftrightarrow \begin{matrix} f_x = 0 \\ f_y = 0 \end{matrix}$ system of equations

③ Decide by looking at $\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$

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Panel 15

What could happen in $\mathbb{R}^2$

$f(x,y) = x^2 + y^2$

① $\nabla f = \langle 2x, 2y \rangle$

② $\nabla f = \langle 0, 0 \rangle$

$(x=0, y=0)$ critical

③ $\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} =$

$\det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4 - 0 = 4$

min

$f_{xx} > 0$

$f(x,y) = -x^2 - y^2$

① $\nabla f = \langle -2x, -2y \rangle$

② critical $(0,0)$

③ $\det \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = 4$

max

$f_{xx} < 0$

$f(x,y) = x^2 - y^2$

① $\nabla f = \langle 2x, -2y \rangle$

② critical $(0,0)$

③ $\det \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = -4$

saddle

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Panel 16

Max / Min Problems

To find max/min of $z = f(x,y)$:

① Find ∇f

② Solve $\nabla f = \vec{0}$ (simultaneous system of equations)

③ Compute $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$ and $D = f_{xx} f_{yy} - (f_{xy})^2$

a) f has min if: $D > 0$, $f_{xx} > 0$

b) f has max if. $D > 0$, $f_{xx} < 0$

c) f has saddle if: $D < 0$

d) no information if: $D = 0$

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Panel 17

Ex: Find and classify the critical points for

$$f(x,y) = x^2 - 2xy + 3y^2 + 4x$$

(1) $f_x = 2x - 2y + 4$

$f_y = -2x + 6y$

(2) $2x - 2y + 4 = 0$

$-2x + 6y = 0$

$4y + 4 = 0$ $y = -1$

$x = -3$

i.e. $(-3, -1)$ is critical

(3) $\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 6 \end{pmatrix}$ $\Rightarrow D = 12 + 4 = 16 > 0$

$f_{xx} = 2 > 0$

$\Rightarrow$ <u>min</u>

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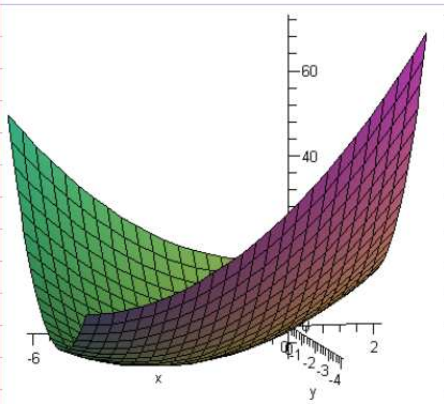
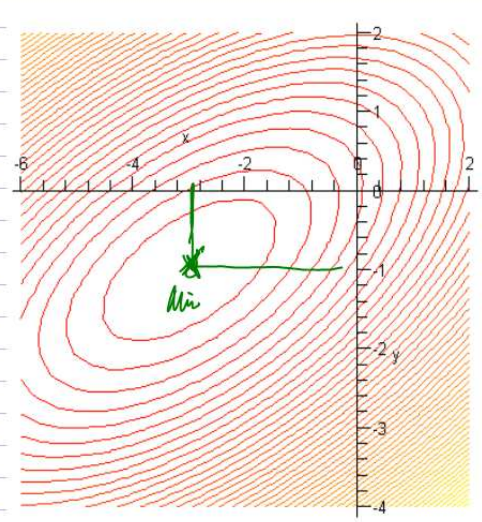
Panel 18

$$f(x,y) = x^2 - 2xy + 3y^2 + 4x$$





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Panel 19

Classify critical points for $f(x,y) = 3x - x^3 - 2y^2$

① $f_x = 3 - 3x^2$ ② $3 - 3x^2 = 0 \iff x^2 = 1$

$f_y = -4y$ $-4y = 0 \iff y = 0$

2 critical points: $(1,0)$ and $(-1,0)$

③ $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} -6x & 0 \\ 0 & -4 \end{pmatrix}$, $D = 24x$

3a) $(1,0)$: $\begin{pmatrix} -6 & 0 \\ 0 & -4 \end{pmatrix}$, $D = 24$ 3b) $(-1,0)$: $\begin{pmatrix} 6 & 0 \\ 0 & -4 \end{pmatrix}$ $D = -24$

max saddle

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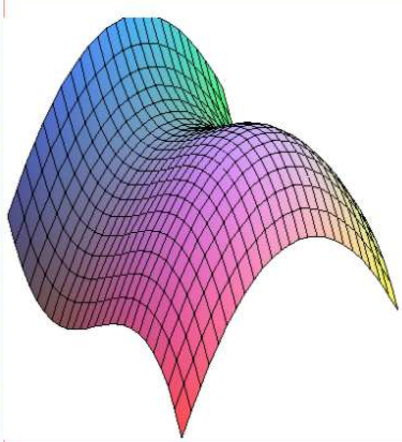
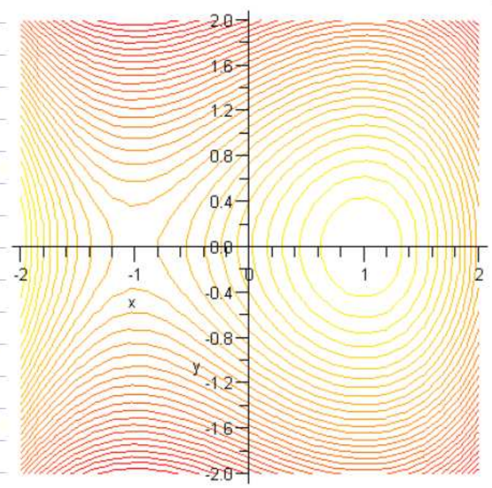
Panel 20

$f(x,y) = 3x - x^3 - 2y^2$





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Panel 21

$f(x,y) = x^4 + y^4 - 4xy + 1$

① $f_x = 4x^3 - 4y$ $\Rightarrow$ ② $x^3 - y = 0 \to y = x^3$

$f_y = 4y^3 - 4x$ $\quad y^3 - x = 0 \longrightarrow x^9 - x = 0$

$x(x^8 - 1) = x(x^4+1)(x^4-1) = x(x^4+1)(x^2+1)(x^2-1) =$

$= x(x^4+1)(x^2+1)(x+1)(x-1) = 0$

$\Rightarrow x = 0, +1, -1$: Critical: $(-1,-1), (0,0), (1,1)$

③ $H = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix} \Rightarrow D = 144x^2y^2 - 16$

at $(-1,-1)$: Min | at $(0,0)$: Saddle | at $(1,1)$: Min

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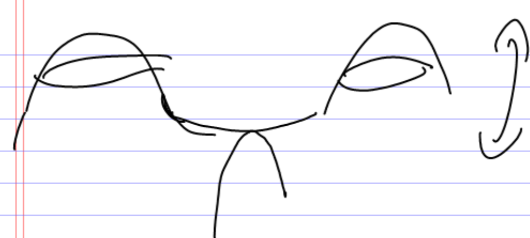
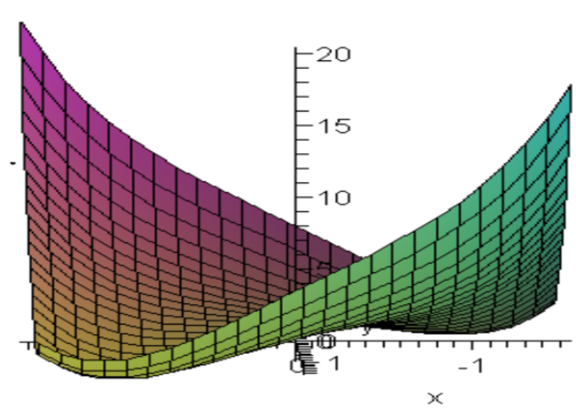
Panel 22





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