Panel 1

Motion in Space

position

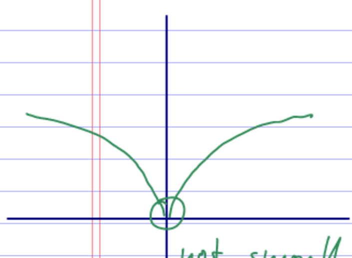
Consider $\vec{r}(t)$ as motion (path) of a particle in space.

$\Rightarrow$ $r'(t)$ is velocity $\vec{v} = \vec{r}'(t)$

$\|r'(t)\|$ is length of $\vec{v}$, i.e. speed $s = \|\vec{v}\| = \|\vec{r}'\|$

$r''(t)$ is acceleration, $\vec{a}(t) = v' = r''$

<u>Ex:</u> Particle moves $r(t) = \langle t^3, t^2 \rangle$ $\qquad T = r'/\|r'\| = \vec{v}/s$



velocity $\vec{v} = \langle 3t^2, 2t \rangle$

speed $s = \sqrt{9t^4 + 4t^2}$

accel. $\vec{a} = \langle 6t, 2 \rangle$

$x = t^3, \; y = t^2$

$\sqrt[3]{x} = t, \; y = x^{2/3}$

1

Panel 2

<u>Ex:</u> A particle starts at $P(1,0,0)$ with initial velocity $\langle 1,-1,1 \rangle$. Acceleration is $\vec{a}(t) = \langle 4t, 6t, 1 \rangle$. Find its velocity and position function.

$$\vec{a}(t) = \langle 4t, 6t, 1 \rangle$$

$$\Rightarrow \vec{v}(t) = \langle 2t^2, 3t^2, t \rangle + C =$$

$$\langle 2t^2, 3t^2, t \rangle + \langle c_1, c_2, c_3 \rangle = \langle 2t^2+1, 3t^2-1, t+1 \rangle$$

Know: $\vec{v}(0) = \langle 1, -1, 1 \rangle = \langle 0, 0, 0 \rangle + \langle c_1, c_2, c_3 \rangle$

$$\Rightarrow C = \langle 1, -1, 1 \rangle$$

$$\vec{r}(t) = \langle \tfrac{2}{3}t^3 + t + \cancel{d_1}^{1}, \; t^3 - t + \cancel{d_2}^{0}, \; \tfrac{1}{2}t^2 + t + \cancel{d_3}^{0} \rangle$$

$$\vec{r}(0) = \langle 1, \; 0, \; 0 \rangle$$

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Panel 3

<u>Ex</u>: An object with mass m moves in a circle with constant angular velocity ω. Find force acting on object and illustrate.

$$\vec{r}(t) = \langle \cos(\omega t), \sin(\omega t) \rangle$$

$$\vec{v}(t) = \langle -\omega \sin(\omega t), \omega \cos(\omega t) \rangle, \quad S = \omega$$

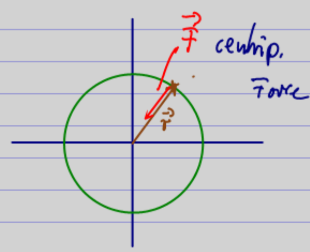
$$\vec{a}(t) = \langle -\omega^2 \cos(\omega t), -\omega^2 \sin(\omega t) \rangle$$

Know $\vec{F} = m\vec{a}$ (Newton's Law)

$$\Rightarrow \vec{F} = -m\omega^2 \langle \cos(\omega t), \sin(\omega t) \rangle$$

$$= -m\omega^2 \vec{r}(t)$$

$\vec{F}$ points to origin



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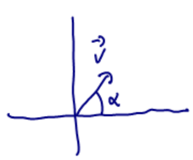
Panel 4

Read book about

ignore friction

- which angle gives max distance



$r(0) = \langle 0, 0 \rangle$

S same, vary α

$\vec{v}_0 = \langle s\cos(\alpha), s\sin(\alpha) \rangle$ const.

$\vec{a} = \langle 0, -g \rangle$

- Kepler's Law of Planetary Motion

$\langle a\cos(\omega t), b\sin(\omega t) \rangle$

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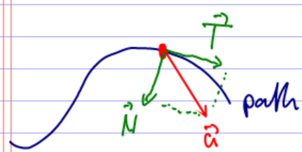
Panel 5

Tangential and Normal Component of Acceleration



acceleration has tangential and normal component.

tang. comp. changes slow down / speeds up

normal. comp. changes direction

$$\vec{a} = \overbrace{a_T \vec{T} + a_N \vec{N}}^{\text{scalar}}$$

Know: $T = \frac{r'}{\|r'\|} = \frac{v}{s} \Rightarrow sT = \vec{v}, \quad \vec{v} = s \cdot T$

$\vec{v} = s \cdot \vec{T} \Rightarrow \vec{a} = s'T + s \cdot T'$

$N = \frac{T'}{\|T'\|}$

$= s'T + s\|T'\| \vec{N}$

$\Rightarrow \|T'\| \cdot N = T'$

$= s'T + s^2 \kappa \vec{N}$

$\kappa = \frac{\|T'\|}{\|r'\|} = \frac{\|T'\|}{s}, \quad \kappa \cdot s = \|T'\|$

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Panel 6

Know: $\vec{a} = s'T + s^2 \kappa \vec{N}$

Want: $\vec{a} = a_T T + a_N N$

$\Rightarrow a_T = s'$ $\qquad a_N = s^2 \kappa$

Also: $\kappa = \frac{\|r' \times r''\|}{\|r'\|^3} = \frac{\|v \times a\|}{s^3} \rightarrow a_N = s^2 \frac{\|v \times a\|}{s^3} = \frac{\|v \times a\|}{s}$

Finally: $\vec{a} = s'T + s^2 \kappa \vec{N} \quad | \cdot \vec{v}$

$\vec{a} \cdot \vec{v} = s'T \cdot v + s^2 \kappa N \cdot \vec{v}$

$\vec{a} \cdot \vec{v} = s'\vec{T} \cdot s\vec{T} + s^2 \kappa N \cdot s\vec{T} =$

$= s's\, T \cdot T + s^3 \kappa \vec{N} \cdot \vec{T}$

$= s's$

$T = \frac{v}{s}$

$sT = v$

$a_T = s' = \frac{a \cdot v}{s}$

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Panel 7

Recall: $\vec{a} = a_T T + a_N N$, $a_T = \frac{\vec{v}\cdot\vec{a}}{s}$, $a_N = \frac{\|\vec{v}\times\vec{a}\|}{s}$

Ex: $\vec{r}(t) = \langle t^2, t^2, t^3 \rangle$ - find tangential and normal components of acceleration

Want: $a_T = \frac{v \cdot a}{s} = \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}}$

$a_N = \frac{\| v \times a \|}{s} = \frac{\sqrt{72t^4}}{\sqrt{8t^2 + 9t^4}}$

$\vec{v} = \langle 2t, 2t, 3t^2 \rangle$

$\vec{v}\cdot\vec{a} = 8t + 18t^3$

$s = \sqrt{8t^2 + 9t^4}$

$\vec{a} = \langle 2, 2, 6t \rangle$

$\vec{v}\times\vec{a} = \begin{vmatrix} i & j & k \\ 2t & 2t & 3t^2 \\ 2 & 2 & 6t \end{vmatrix} = \langle 6t^2, -6t^2, 0 \rangle$

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Panel 8

Quiz 4

Suppose $\vec{r}(t) = \langle t^2, 2, t \rangle$ is a vector-valued function (aka space curve), representing the position of a particle. Find the following:

1. The velocity at $P(0,0,0)$ ✓
2. The speed at $P(0,0,0)$ ✓
3. The acceleration at $P(0,0,0)$ ✓
4. The unit tangent $\vec{T}(t)$ at $P(0,0,0)$
5. The unit normal vector $\vec{N}(t)$ at $P(0,0,0)$ Hard
6. The bi-normal vector $\vec{B}(t)$ at $P(0,0,0)$
7. The curvature k at $P(0,0,0)$
8. The tangential component of the acceleration a_T at $P(0,0,0)$ ✓
9. The normal component of the acceleration a_N at $P(0,0,0)$ ✓
10. The osculating plane at $P(0,0,0)$
11. The osculating circle at $P(0,0,0)$

Done with #4

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Panel 9

Functions of Several Variables

Know: $f: \mathbb{R} \to \mathbb{R}$ e.g. $f(x) = x^2$

$r: \mathbb{R} \to \mathbb{R}^2$ e.g. $r(t) = (t, t^2)$

$r: \mathbb{R} \to \mathbb{R}^3$ e.g. $r(t) = (\cos(t), \sin(t), t)$

Next: $f: \mathbb{R}^2 \to \mathbb{R}$ $f(x,y) = x^2 + y^2$

$f: \mathbb{R}^3 \to \mathbb{R}$ $f(x,y,z) = xyz$

(Later: $f: \mathbb{R}^2 \to \mathbb{R}^2$)

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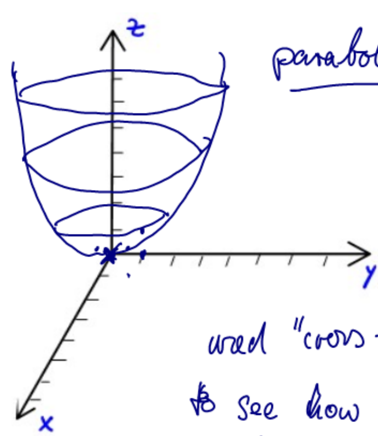
Panel 10

Def: A function of 2 variables is a rule that assigns to every pair (x,y) in a set D a unique real number denoted by $f(x,y)$.

How to visualize $f(x,y) = x^2 + y^2 = z$

x	y	$f(x,y) = z$
→ 0	0	0 ←
→ 0	1	1
→ 1	0	1
1	1	2
1	2	5
2	1	5
2	2	8
⋮	⋮	⋮



paraboloid

used "cross-sections" to see how it could look

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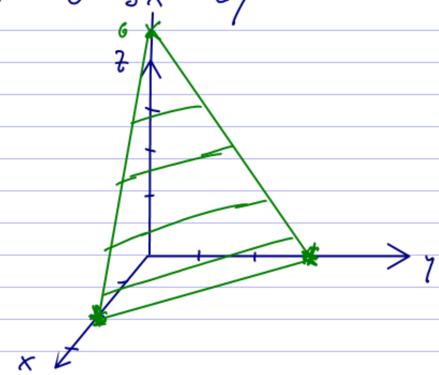
Panel 11

Ex: $f(x,y) = 6 - 3x - 2y$

$z = 6 - 3x - 2y$ ~ plane!



Cross-sections: $x = 0: \quad z = 6 - 2y$ $(y=0,\ z=6)$ $(z=0 \to y=3)$

$y = 0: \quad z = 6 - 3x$

$z = 0: \quad 0 = 6 - 3x - 2y \Rightarrow 3x + 2y = 6$

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Panel 12

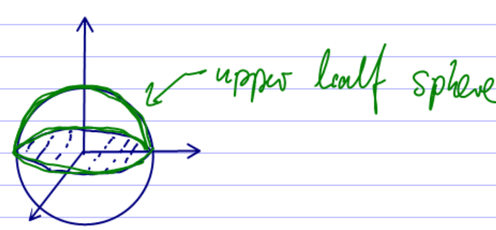
Ex: $f(x,y) = \sqrt{9 - x^2 - y^2}$, z always ≥ 0

$z = \sqrt{9 - x^2 - y^2} \Leftrightarrow z^2 + x^2 + y^2 = 9$

$x = 0: \quad z = \sqrt{9 - y^2} \Leftrightarrow z^2 + y^2 = 9$

$y = 0: \quad z = \sqrt{9 - x^2} \Leftrightarrow z^2 + x^2 = 9$

$z = 0: \quad 0 = \sqrt{9 - x^2 - y^2} \Leftrightarrow x^2 + y^2 = 9$



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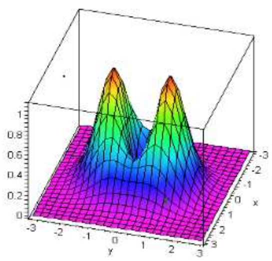
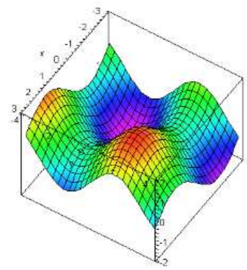
Panel 13

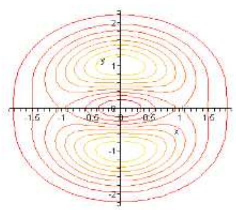
$f(x,y) = (x^2+3y^2)e^{-x^2-y^2}$

$f(x,y) = \sin(x) + \sin(y)$

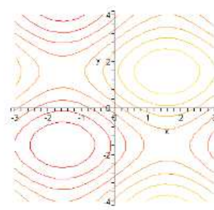




level-curves are curves at fixed height



contour plot



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Panel 14

Of course I used Maple to generate these plots

```
> plot3d((x^2+3*y^2)*exp(-x^2-y^2),x=-3..3,y=-4..4);
> plot3d(sin(x)+sin(y),x=-3..3,y=-4..4);
> with(plots);
> contourplot((x^2+3*y^2)*exp(-x^2-y^2),x=-3..3,y=-4..4);
> contourplot(sin(x)+sin(y),x=-3..3,y=-4..4);
> 
```

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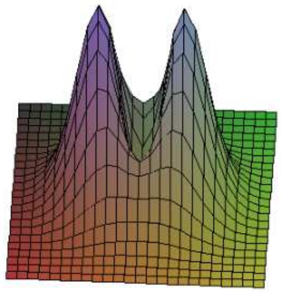
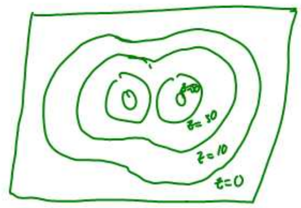
Panel 15





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Panel 16

<u>Ex:</u> Sketch level curves of $h(x,y) = 4x^2 + y^2$

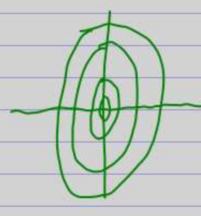
$4x^2 + y^2 = 1$

$4x^2 + y^2 = 2$

$\vdots$

$4x^2 + y^2 = k$

<u>elipses</u>



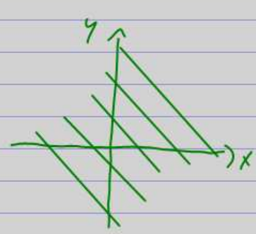
elipsoid

<u>Ex:</u> Sketch level curves for $f(x,y) = 6 - 3x - 2y$

$6 - 3x - 2y = k$

$6 - k = 3x + 2y$

$\frac{6-k}{2} - \frac{3}{2}x = y$



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Panel 17

Limits and Continuity

$\lim_{(x,y)\to(0,0)} x^2+y^2 = 0$ ↑ clearly but $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ = l'Hospital DOES NOT WORK !!!

Bad news: Limits in $\mathbb{R}^2$ are

* way *

more complicated than

in $\mathbb{R}$

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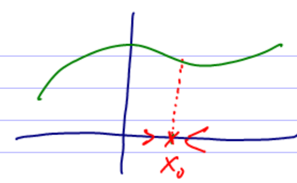
Panel 18

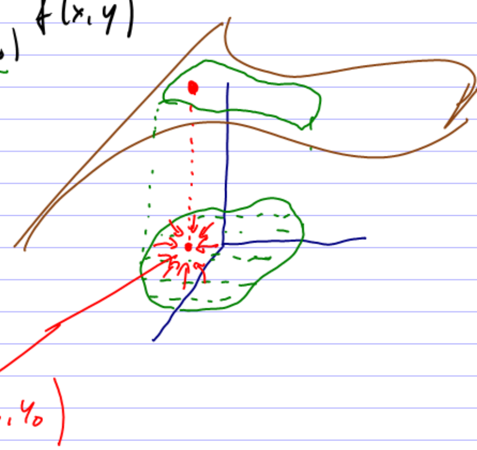
In $\mathbb{R}$: $\lim_{x\to x_0} f(x)$

2 ways to get close to x_0



In $\mathbb{R}^2$: $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$

infinitely many ways to approach (x_0,y_0)



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Panel 19

Ex: Does $\lim\limits_{(x,y)\to(1,1)} \dfrac{xy}{x^2+y^2}$ exist?

$$\frac{1}{2}$$

Ex: Does $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ exist? $\frac{0}{0}$ unknown? Does not exist

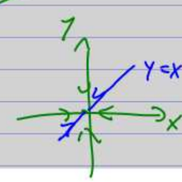
But: approach $(0,0)$ along x-axis:

$y=0$: $\lim\limits_{x\to 0} \dfrac{x\cdot 0}{x^2+0^2} = 0$

y-axis: $x=0$: $\lim\limits_{y\to 0} \dfrac{0\cdot y}{0^2+y^2} = 0$

diagonal: $y=x$: $\lim\limits_{x\to 0} \dfrac{x\cdot x}{x^2+x^2} = \frac{1}{2}$

different



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Panel 20

Ex: Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{3xy^2}{x^2+y^4}$ if it exists $\boxed{\text{Not exist}}$

$x=0$: $\lim \dfrac{3\cdot 0}{y^4} = 0$

$y=0$: $\lim \dfrac{3\cdot 0}{x^2} = 0$

$x=y$: $\lim \dfrac{3x^3}{x^2+x^4} = \lim \dfrac{x^3}{x^2(1+x^2)} = \lim\limits_{x\to 0} \dfrac{x}{1+x^2} = 0$

$y=x^2$: $\lim \dfrac{3x^5}{x^2+x^8} = 0$

$x=y^2$: $\lim \dfrac{3y^4}{y^4+y^4} = \frac{3}{2}$

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Panel 21

Ex: $\lim\limits_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

Stuck!

Maybe limit does exist?

✓ $x=0$: $\lim = 0$

✓ $y=0$: $\lim = 0$

✓ $x=y$: $\lim \frac{3x^3}{2x^2} = 0$

✓ $x=y^2$: $\lim\limits_{y\to 0} \frac{3y^5}{y^4+y^2} = 0$

✓ $x^2=y$

✓ $y=\sin(x)$: $\lim\limits_{x\to 0} \frac{3x^2\sin(x)}{x^2+\sin^2(x)} = 0$

$\cancel{y} = \cos(x)$

$y = x\cos(x)$

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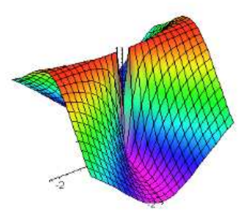
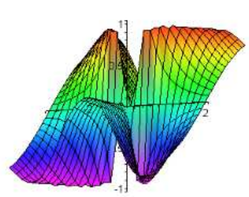
Panel 22

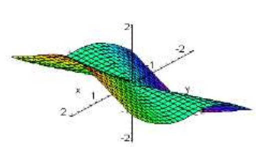
$f(x,y) = \frac{x^2}{x^2+y^2}$



$f(x,y) = \frac{2x^2y}{x^4+y^2}$



$f(x,y) = \frac{x^3}{x^2+y^2}$



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