Panel 1

Warm-up Question:

A Is the line l(t) = <1 + t, 2 + 2t, 1 + 3t> contained in the plane 12x -3y - 2z = 4 ?

1. Find 2 points on the line:

$t=0: \; P(1,2,1) \;\Rightarrow\; 12\cdot 1 - 3\cdot 2 - 2\cdot 1 = 4$ ✓

$t=1: \; Q(2,4,4) \;\Rightarrow\; 12\cdot 2 - 3\cdot 4 - 2\cdot 4 = 4$ ✓

YES

B Find a line that is contained in that plane

Pick $z = 0$: $12x - 3y = 4$, $y = 4x - 4/3$

Let $x = t$, $y = 4t - 4/3$

$l(t) = \langle t, 4t - 3/3, 0 \rangle$

1

Panel 2

Decide which points are part of the given line and plane (check appropriate boxes):

-1 0 2 t=-1

Line: **l(t)** = <t, 1 + t, 1 - t>.

P(1, 1, 2)

Plane: 2x - 4y + z = 0

-2 0 2

[ ] is part of the line → $(1,1,2) = \langle t, 1+t, 1-t \rangle$ ONE t

[X] is part of the plane

$\left.\begin{array}{l} 1 = t \\ 1 = 1+t \\ 2 = 1-t \end{array}\right\}$ no solution

✓ ✓ ✓

Q(-1, 0, 2)

[X] is part of the line

[X] is part of the plane

other point on line is:

$t=0: \; (0,1,1)$

2

Panel 3

Last time

Dot product ✓

Cross product: $\langle 1,0,3\rangle \times \langle -2,1,2\rangle$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & 3 \\ -2 & 1 & 2 \end{vmatrix} = \langle 0-3, -(2+6), 1-0 \rangle$$

→ Property of $v \times w$: $(v\times w)\cdot v = (v \times w)\cdot w = 0$

Equation of line: $\ell(t) = r_0 + t\vec{v} = \langle x_0 + tv_1, y_0 + tv_2, z_0 + tv_3\rangle$

Equation of plane: ?

HW.

Page 849: #35, 38, 40, 45, 47, 52
Read pages 855-856
Page 856: #2, 5, 8, 9, 13, 14, 15, 25 (a), 35, 36, 41, 44, 45
Page 865: #1, 2, 3, 4, 13, 14, 19, 20

3

Panel 4

856 # 36 $\langle 1,-1,1\rangle, \langle 0,4,4\rangle$

Find 2 unit vectors perp to both.

a) find a perp vector.

$$\begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 0 & 4 & 4 \end{vmatrix} = \langle -8, -(4), 4 \rangle.$$

b) unit vector: $\frac{1}{\sqrt{96}}\langle -8, -4, 4\rangle$

c) other: $\frac{1}{\sqrt{96}}\langle 8, 4, -4\rangle$

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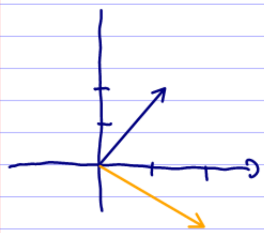
Panel 5

in $\mathbb{R}^2$: $\langle 1,2 \rangle$ find 2 vectors

perp. to $\langle 1,2 \rangle \cdot \langle 2,-1 \rangle = 0$



perp. to $\langle 2,1,3 \rangle$

a) $\langle 0,3,-1 \rangle$ or

$\langle -3,0,2 \rangle$

perp. to $\langle 1,2,3,4 \rangle \cdot \langle 0,0,-4,3 \rangle$

Tricky: 1 vector perp. to 2 others: cross prod.

5

Panel 6

Quiz # 3

① Suppose $v = \langle -1,3,2 \rangle$ and $\vec{w} = \langle 1,0,2 \rangle$. Find

a) $\vec{v} \cdot \vec{w}$

b) $\vec{v} \times \vec{w}$

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Panel 7

Quiz #3

(2) Suppose $\vec{v} = \langle -1, 3, 2 \rangle$ and $\vec{w} = \langle 1, 0, 2 \rangle$. Find

a) $(\vec{v} \times \vec{w}) \cdot \vec{v} = 0$

b) $\text{proj}_{\vec{v}}(\vec{w})$

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Panel 8

Quiz #3

(3) Find the equation of the line through the points $P(1,0,2)$ and $Q(-1,2,0)$

8

Panel 9

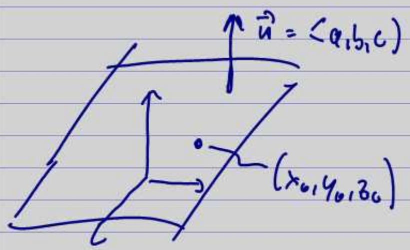
Scalar equation of Plane through $P(x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$ is $\underline{a}(x - x_0) + \underline{b}(y - y_0) + \underline{c}(z - z_0) = 0$

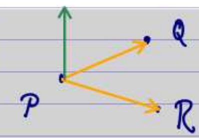
Ex: Plane through $P(1,3,2)$, $Q(3,-1,6)$, and $R(5,2,0)$

HW

- $ax + by + cz + d = 0$
- $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$
- $ax + by + cz = D$

$\vec{n} = \langle a, b, c \rangle$





$\vec{PQ} = \langle \qquad \rangle$

$\vec{PR} = \langle \qquad \rangle$

$\vec{PQ} \times \vec{PR} = \langle a, b, c \rangle$

$\underset{a}{\underline{\quad}}(x - 1) + \underset{b}{\underline{\quad}}(y - 3) + \underset{c}{\underline{\quad}}(z - 2) = 0$

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Panel 10

Find the equation of the plane through the point P(1, -1, 2) containing the vectors **v** = <1, 0, 3> and **w** = <-1, 1, 0>.

(1) $\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 1 & 0 & 3 \\ -1 & 1 & 0 \end{vmatrix} = \langle -3, -3, 1 \rangle$

(2) $-3(x - 1) - 3(y + 1) + 1(z - 2) = 0$

$-3x + 3 - 3y - 3 + z - 2 = 0$

$-3x - 3y + z = 2$

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Panel 11

Scalar equation of Plane through $P(x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$ is $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

Ex: Find angle between $x+y+z=1$ and $x-2y+3z=1$

Def. the angle between 2 planes is the angle between their normal vectors

Plane $x+y+z=1 \Rightarrow n_1 = \langle 1,1,1 \rangle$

Plane $x-2y+3z=1 \Rightarrow n_2 = \langle 1,-2,3 \rangle$

$$\cos(\alpha) = \frac{1-2+3}{\sqrt{3}\,\sqrt{14}} = 2/\sqrt{52} \qquad ($$

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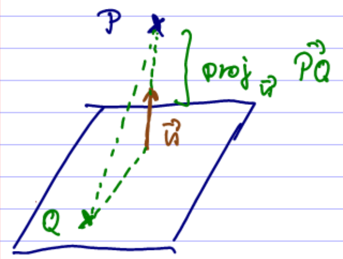
Panel 12

Distance between Point $P(x_1, y_1, z_1)$ and Plane $ax+by+cz+d=0$



Ex: Find distance of $10x+2y-2z=5$ to $P(0,0,0)$

① Find extra point on plane $Q(\frac{1}{2}, 0, 0)$

② normal vector $\vec{n} = \langle 10, 2, -2 \rangle$

③ distance $= \left|\text{proj}_{\vec{n}} \vec{PQ}\right| = \left|\frac{\vec{PQ}\cdot\vec{n}}{\|\vec{n}\|}\right| \quad \frac{\vec{n}}{\|\vec{n}\|} = \frac{\langle \frac{1}{2},0,0\rangle \cdot \langle 10,2,-2\rangle}{\sqrt{108}} = 5/\sqrt{108}$

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Panel 13

Distance between $Q(x_1, y_1, z_1)$ and plane $ax + by + cz = d$:

① Find arbitrary point $P(x_0, y_0, z_0)$ on plane

② Find vector $\vec{PQ}$

③ Compute distance $D = \frac{|\vec{PQ} \cdot \vec{n}|}{\|\vec{n}\|}$

Ex: Find distance between $(2,8,5)$ and $x - 2y - 2z = 1$

~~point 1: (2,8,5) P~~

~~point 2: (1,0,0) Q~~

~~> vector $QP = \langle 1,8,5 \rangle$~~

~~$\text{dist.} = \left| \frac{QP \cdot \vec{n}}{\|\vec{n}\|} \right| = \left| \frac{\langle 1,8,5\rangle \cdot \langle 1,-2,-2\rangle}{} \right| = \left| \frac{1 - 16 \ldots}{\sqrt{10}} \right| \ldots 10 \ldots 0$~~

~~$= \left| -\frac{20}{\sqrt{10}} \right| = \frac{20}{\sqrt{10}}$~~

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13

Panel 14

Distance between $Q(x_1, y_1, z_1)$ and plane $ax + by + cz = d$:

① Find arbitrary point $P(x_0, y_0, z_0)$ on plane

② Find vector $\vec{PQ}$

③ Compute distance $D = \frac{|\vec{PQ} \cdot \vec{n}|}{\|\vec{n}\|}$

Ex: Find distance between $(2,8,5)$ and $x - 2y - 2z = 1$

together

14

Panel 15

Find distance between

a) $10x + 2y - 2z = 5$ and $x + y + z = 1$

b) $10x + 2y - 2z = 5$ and $5x + y - z = 1$

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Panel 16

Find distance between

a) $10x + 2y - 2z = 5$ and $x + y + z = 1$

b) ① $10x + 2y - 2z = 5$ and ② $5x + y - z = 1$

a) $n_1 = \langle 10, 2, -2 \rangle$, $n_2 = \langle 1, 1, 1 \rangle$

n_1 and n_2 are not parallel $\Rightarrow$ planes intersect $\Rightarrow$ distance is zero!

b) $n_1 = \langle 10, 2, -2 \rangle$ $n_2 = \langle 5, 1, -1 \rangle$ are parallel.

point on plane 1: $P(\frac{1}{2}, 0, 0)$, on plane 2: $Q(0, 1, 0)$

$$d = \frac{|\vec{PQ} \cdot \vec{n}_1|}{\|\vec{n}_1\|} = \frac{\langle \frac{1}{2}, -1, 0 \rangle \cdot \langle 5, 1, -1 \rangle}{\sqrt{27}} = \frac{5\frac{1}{2} - 1}{\sqrt{27}} = \frac{3}{2\sqrt{27}}$$

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Panel 17

Ex: Distance between $l_1(t) = \langle 1+t, -2+3t, 4-t \rangle$ and
$l_2(s) = \langle 2s, 3+s, -3+4s \rangle$

Already know lines are skew (not parallel, no intersection).

Check the book!

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Panel 18

Rest of chapter 13 deals with

→ Quadric Surfaces — skip (read)

→ Cylindrical / spherical Coordinates — when needed

Key Concepts we did cover

$\mathbb{R}^3$, sheets, spheres

Vectors, length, add/subtract

Dot + Cross products

Lines + Planes

Distances

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Panel 19

Chapter 14: Space curves $\vec{r}: \mathbb{R} \to \mathbb{R}^3$ (or $\mathbb{R}^2$)

Def: $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is a vector-valued function with component functions f, g, and h

Many concepts work as they should: If $\vec{r}(t)$ is vector-valued function then

Limit: $\lim_{t \to t_0} \vec{r}(t) = \langle \lim_{t \to t_0} f(t), \lim_{t \to t_0} g(t), \lim_{t \to t_0} h(t) \rangle$

Derivative: $\vec{r}\,'(t) = \frac{d}{dt} \vec{r}(t) = \langle f'(t), g'(t), h'(t) \rangle$

Integral: $\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$

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Panel 20

Ex: $\vec{r}(t) = \left\langle \frac{t^2-9}{t-3}, \frac{\sin(t)}{t}, \frac{\cos(t)-1}{t} \right\rangle$

Find: $\lim_{t \to 0} \vec{r}(t) = \left\langle \lim_{t \to 0} \frac{t^2-9}{t-3}, \lim_{t \to 0} \frac{\sin(t)}{t}, \lim_{t \to 0} \frac{\cos(t)-1}{t} \right\rangle = \langle 3, 1, 0 \rangle$

$\frac{\sin(x)}{x}$

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Panel 21

The problem with vector-valued functions is to visualize them, and interprate the deriv. + integrals:

Ex: $\vec{r}(t) = \langle 1+t, 2+5t, -1+6t \rangle$ - describe graph

some line

Ex: $\vec{r}(t) = \langle \underbrace{\cos t, \sin t,} t \rangle$ - describe graph

$x = \cos(t)$
$y = \sin(t)$
$z = t$

Slinky around z-axis

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Panel 22

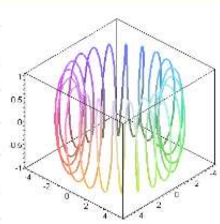
Sketch graph of

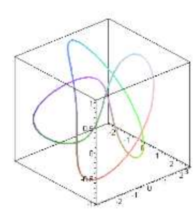
$r_1(t) = \langle (4 + \sin(20t)) \cos(t), (4 + \sin(20t)) \sin(t), \cos(20t) \rangle$

$r_2(t) = \langle (2 + \cos(1.5t)) \cos t, (2 + \cos(1.5t)) \sin(t) \quad \sin(1.5t) \rangle$

(t, t^2, t^3)

```
> with(plots);
> spacecurve( [(4+sin(20*t))*cos(t), (4+sin(20*t))*sin(t), cos(20*t)], t=0..2*Pi, numpoints=500);
> spacecurve( [(2+cos(1.5*t))*cos(t), (2+cos(1.5*t))*sin(t), sin(1.5*t)], t=0..4*Pi, numpoints=500);
> spacecurve([t, t^2, t^3], t=0..2);
```





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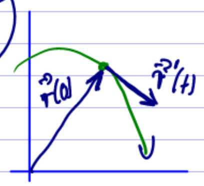
Panel 23

Derivatives of Space Curves aka Vector-valued functions

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ and f, g, h are differentiable

then $\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$

$\mathbb{R}^2$



Ex: $\vec{r}(t) = \langle 1+t^3, te^{-t}, \sin(2t) \rangle$

Find $\vec{r}(0)$ and $\vec{r}\,'(0)$.

Compute $\vec{r}(0) \cdot \vec{r}\,'(0)$

$\vec{r}(0) = \langle 1, 0, 0 \rangle$

$\vec{r}\,'(t) = \langle 3t^2, e^{-t} - te^{-t}, 2\cos(2t) \rangle$

$\vec{r}\,'(0) = \langle 0, 1, 2 \rangle \Rightarrow \vec{r}(0) \cdot \vec{r}\,'(0) = 0$

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Panel 24

Ex. Find equation of tangent line to $r(t) = \langle 2\cos t, \sin t, t \rangle$ at the point $P(0, 1, \pi/2)$

Need $\ell(t) = \vec{r}_0 + t\vec{v} = (x_0 + tv_1, y_0 + tv_2, z_0 + tv_3)$

$P(0, 1, \pi/2)$

$\vec{v}$: $\vec{r}\,'(t) = \langle -2\sin(t), \cos(t), 1 \rangle$, $t = \pi/2$

which t s.t. $(2\cos(t), \sin(t), t) = \langle 0, 1, \pi/2 \rangle$

$\vec{r}\,'(\pi/2) = \underline{\langle -2, 0, 1 \rangle}$

$\ell(t) = (0, 1, \pi/2) + t(-2, 0, 1) = \langle -2t, 1, t + \pi/2 \rangle$

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Panel 25

Arc Length

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ then $L = \int_a^b |r'(t)|\, dt$ is the length of the space curve.

nice formula, easy to remember

Note: $\int_a^b |r'(t)|\, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$

Ex: Find length of $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$, $t = 0$ to 2π

$r(t) = \langle \cos(t), \sin(t) \rangle$ $\boxed{t = 0 \text{ to } 2\pi}$

circle

$r'(t) = \langle -\sin(t), \cos(t) \rangle$

$|r'(t)| = \sqrt{\sin^2(t) + \cos^2(t)} = 1$

$L = \int_0^{2\pi} |r'(t)|\, dt = \int_0^{2\pi} 1\, dt = 2\pi$

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Panel 26

$r(t) = (x(t), y(t))$

$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^3}\, dt$

length $\approx \sum \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt =$

$\lim_{dt \to 0}$

$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt =$

$= \int_a^b |r'(t)|\, dt$

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Panel 27

Ex: Set up an integral for length of space curve
$\vec{r}(t) = \langle t, t^2, t^3 \rangle$ as $t = 0$ to 1.

$r'(t) = \langle 1, 2t, 3t^2 \rangle$

$|r'(t)| = \sqrt{1 + (2t)^2 + (3t^2)^2} = \sqrt{1+4t^2+9t^4}$

$L = \int_0^1 \sqrt{1+4t^2+9}$

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