Panel 1

Last Time:

Review of Calc 1+2

Intro to $\mathbb{R}^3$

Point, distance

Spheres + sheets as 3D objects

Maple

→ (Vectors)

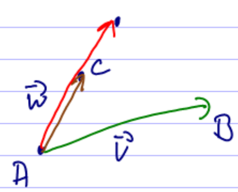
Homework is section 13.1, page 833:

#6, 8, 9, 10, 11, 14, 15, 16, 23, 24, 27, 30, 32, 33, 34

1

Panel 2

#9) $A(5,1,3)$, $B(7,9,-1)$, $C(1,-15,11)$ on a line?

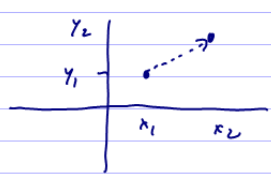


They are on a line if vectors $\vec{v} = k\vec{w}$, some k

$\vec{v} = \vec{AB} = \langle 7-5, 9-1, -1-3 \rangle = \langle 2, 8, -4 \rangle$, $\vec{AC} = \langle -4, -16, 8 \rangle$

Recall: $A = (x_1, y_1)$, $B = (x_2, y_2)$

then $\vec{AB} = \langle x_2 - x_1, y_2 - y_1 \rangle$



dist. between O and A: $\sqrt{x_1^2 + y_1^2}$

dist. between A and B: $\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

2

Panel 3

$\vec{v} = \vec{AB} = \langle 7-5, 9-1, -1-3 \rangle = \underline{\langle 2, 8, -4 \rangle}$, $\vec{AC} = \underline{\langle -4, -16, 8 \rangle}$

$\vec{v} = \langle 2, 8, -4 \rangle$

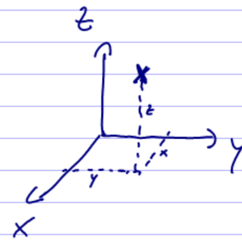
$\vec{w} = \langle -4, -16, 8 \rangle$

Is $\vec{v} = k\vec{w}$? Yes $k = -2$

$\boxed{\#10}$ $P(3, 7, -5)$ distance to

a) xy-plane ✓

d) x-axis ✓



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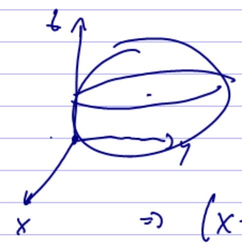
Panel 4

$\boxed{\#14}$ Sphere through origin, center $\underline{(1, 2, 3)}$



$\underline{\text{Sphere}}$: $(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2$

$\Rightarrow (x-1)^2 + (y-2)^2 + (z-3)^2 = r^2$

$(0-1)^2 + (0-2)^2 + (0-3)^2 = r^2$

$1 + 4 + 9 = r^2 \Rightarrow \underline{\underline{r = \sqrt{14}}}$

$\boxed{\#16}$ $x^2 + y^2 + z^2 = 4x - 2y$ center $(2, -1, 0)$

$x^2 - 4x + y^2 + 2y + z^2 = 0$ radius $\sqrt{5}$

$(x-2)^2 + (y+1)^2 + z^2 = 4 + 1$

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Panel 5

Quiz 1

(0) What is the definition of "derivative"?

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Panel 6

Quiz 1

(1) Find the distance between the origin and the point $P(2,1,-1)$

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Panel 7

Quiz 1

② The equation $x^2 + y^2 + z^2 - 6y = 7$ represents a sphere. Find the center and radius of that sphere.

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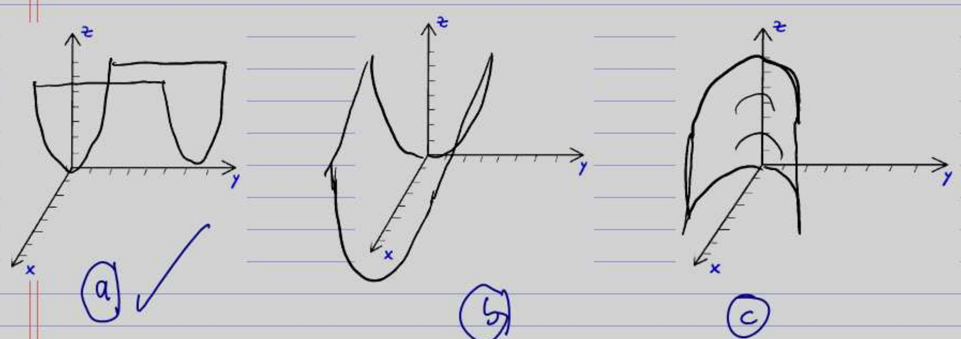
Panel 8

Quiz 1

③ Circle the graph that represents $z = x^2$?



When you are done, submit 4 panels!

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Panel 9

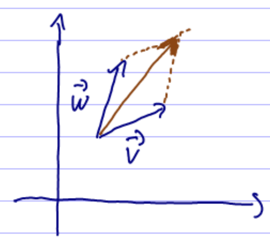
Vectors:

Ex: $P(1,-2)$ and $Q(3,3)$ are two points. What is:

$\langle 1,-2\rangle$ is vector

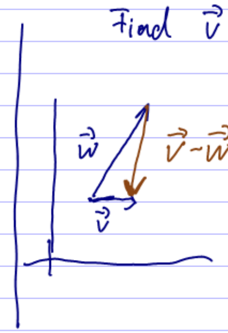
$\vec v = \vec{PQ} = \langle 3-1, 3+2\rangle = \langle 2,5\rangle$

$\Rightarrow \vec v + 2\langle 1,-2\rangle = \langle 2,5\rangle + \langle 2,-4\rangle = \langle 4,1\rangle$



Find $\vec v + \vec w$

Find $\vec v - \vec w$



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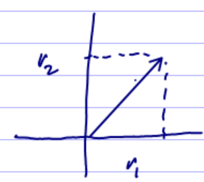
Panel 10

Vectors, Continued

Vector from $P(x_1,y_1)$ to $Q(x_2,y_2)$ is:

$\vec v = \langle x_2-x_1, y_2-y_1\rangle$



Length, or Norm, of vector $\vec v = \langle v_1, v_2\rangle$ is:

$\|\vec v\| = |\vec v| = \sqrt{v_1^2+v_2^2}$

$\cancel{\|(3,4)\|}$ no meaning

Examples: $\|\langle 3,4\rangle\| = \sqrt{9+16} = \sqrt{25} = 5$

$1/\|\langle -1,2\rangle\| = 1/\sqrt{5}$

$\cancel{\frac{\vec v}{\vec w}}$ no meaning

$\vec v = \langle -4,3\rangle$, find $\left\|\frac{1}{\|v\|}\cdot\vec v\right\| = \left\|\frac{1}{5}\langle -4,3\rangle\right\| = \left\|\left\langle -\frac{4}{5}, \frac{3}{5}\right\rangle\right\| = 1$

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Panel 11

Def: A unit vector: is a vector $\vec{v}$ of length $\|\vec{v}\| = 1$

Ex: Which one is a unit vector

$\cancel{\langle \frac{1}{2}, \frac{3}{4} \rangle}$ or $\langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$ ✓

Thm: If $\vec{v}$ is any (non-zero) vector, then $\underbrace{\frac{1}{\|\vec{v}\|} \cdot \vec{v}}$ is a unit vector, same dir. as $\vec{v}$ but length 1

There are special unit vectors:

(x-axis) $\vec{i} = \langle 1, 0, 0 \rangle$
(y-axis) $\vec{j} = \langle 0, 1, 0 \rangle$
(z-axis $\vec{k} = \langle 0, 0, 1 \rangle$

basic unit vectors

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Panel 12

Ex: If $P = (3,1,2)$ and $Q = (1,1,4)$ find

$\vec{PQ} : \langle -2, 0, 2 \rangle$

$-2\vec{i} + 2\vec{k} = -2\langle 1,0,0 \rangle + 2 \cdot \langle 0,0,1 \rangle = \langle -2, 0, 2 \rangle$

$\vec{v} = \langle 2, 0, -2 \rangle = 2\hat{i} + 0\hat{j} - 2\hat{k}$

Every vector can be expressed in

a) component notation

b) sum of multiples of basic unit vectors

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Panel 13

Ex: $5i + 7j + 9k + 3i + 7k - j = 8i + 6j + 16k$

$$\langle 5, 7, 9 \rangle + \langle 3, -1, 7 \rangle = \langle 8, 6, 16 \rangle$$

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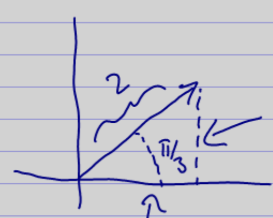
Panel 14

Ex: Find a vector of length 2 that makes an angle of $\pi/3$ with positive axis.

Want $\langle x, y \rangle$ s.t. $\| \langle x, y \rangle \| = 2$ and angle is $\pi/3$



$y = 2\sin(\pi/3)$

$x = 2\cos(\pi/3)$

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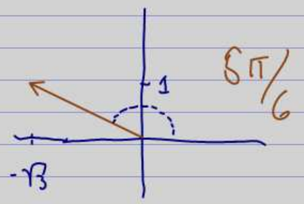
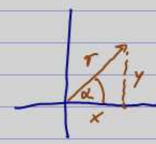
Panel 15

Ex: Suppose $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$. Find the angle that $v = -\sqrt{3}\,i + j$ makes with positive x-axis.

$v = \langle -\sqrt{3}, 1 \rangle$



Recall:



$y = r\sin(\alpha)$

$x = r\cos(\alpha)$

$\frac{y}{x} = \tan(\alpha)$

$\alpha = \arctan(y/x)$

$\alpha = \arctan(-1/\sqrt{3})$

$\alpha = -\pi/6$

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Panel 16

Multiplying Vectors

You think $\langle 3, 2 \rangle \bullet \langle 5, 6 \rangle = \cancel{\langle 15, 12 \rangle} = 27$

Mult. of vectors is counter-intuitive.

Dot Product:

If $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$ then

$\vec{v} \underset{\text{"dot"}}{\bullet} \vec{w} = v_1 \cdot w_1 + v_2 \cdot w_2$

If $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ then

$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$

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Panel 17

Examples of Dot Product

① $\langle 3,5\rangle \cdot \langle -1,2\rangle = 7$

② $\langle 2,3\rangle \cdot \langle -3,2\rangle = 0$

③ $\langle 1,-3,4\rangle \cdot \langle 1,5,2\rangle = -6$

$1 - 15 + 8 = -6$

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Panel 18

Properties of Dot Product

a) $\vec{a} \cdot \vec{a} = \|a\|^2$ ✓ !

b) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ ⇐

c) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ ⇐

Proof of a)

$\mathbb{R}^2$: $\vec{a} = \langle a_1, a_2\rangle \Rightarrow \vec{a} \cdot \vec{a} = \langle a_1, a_2\rangle \cdot \langle a_1, a_2\rangle = a_1^2 + a_2^2$

$\|\vec{a}\|^2 = \left(\sqrt{a_1^2 + a^2}\right)^2 = a_1^2 + a_2^2$ #

Proof of b)

$\langle a_1, a_2\rangle \cdot \left(\langle b_1, b_2\rangle + \langle c_1, c_2\rangle\right) =$

$\langle a_1, a_2\rangle \cdot \langle b_1 + c_1, b_2 + c_2\rangle = a_1 \cdot (b_1 + c_1) + a_2 \cdot (b_2 + c_2) =$

$= a_1 b_1 + a_1 c_1 + a_2 b_2 + a_2 c_2$

$= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

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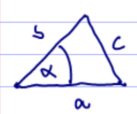
Panel 19

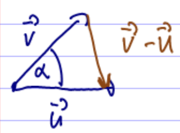
Theorem: If $\vec{u}$ and $\vec{v}$ are non-zero vectors in $\mathbb{R}^2$ then

$$\frac{\vec{u}\cdot\vec{v}}{\|u\|\cdot\|v\|} = \cos(\alpha) \quad , \alpha = \text{angle between } \vec{u}, \vec{v}$$

Recall



law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\alpha$



$$\Rightarrow \quad \|\vec{v}-\vec{u}\|^2 = \|\vec{u}\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos(\alpha)$$

$$(v-u)\cdot(v-u) = u\cdot u + v\cdot v - 2\|u\|\|v\|\cos(\alpha)$$

$$\cancel{vv} - vu - u\cdot v + \cancel{u\cdot u} = \cancel{u\cdot u} + \cancel{v\cdot v} - 2\|u\|\|v\|\cos(\alpha)$$

$$\cancel{-2}\, u\cdot v = \cancel{-2}\|u\|\|v\|\cos(\alpha)$$

$$\frac{u\cdot v}{\|u\|\|v\|} = \cos(\alpha)$$

Panel 20

Ex: Find angle between $u = i - 2j + 2k = \langle 1,-2,2\rangle$ and

a) $v = -3i + 6j + 2k = \langle -3, 6, 2\rangle$

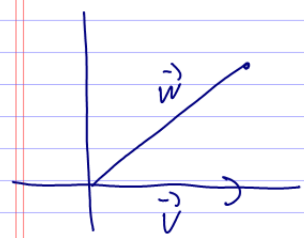
$$\cos(\alpha) = \frac{u\cdot v}{\|u\|\cdot\|v\|} = \frac{\langle 1,-2,2\rangle\cdot\langle -3,6,2\rangle}{\sqrt{9}\cdot\sqrt{49}} = \frac{-3-12+4}{21} = \frac{-11}{21}$$

b) $w = 2i + 7j + 6k = \langle 2, 7, 6\rangle$

$\cos(\alpha) = 0$ because $\vec{u}\cdot\vec{w} = 0$

$\Rightarrow \alpha = \pi/2$ or $3\pi/2$

Panel 21

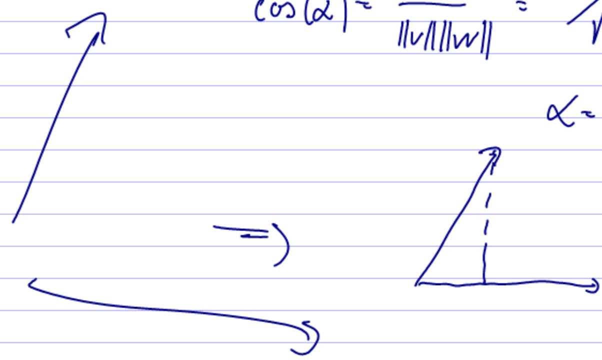


$\vec{w} = \langle 1, 1 \rangle$

$\vec{v} = \langle 1, 0 \rangle$

$$\cos(\alpha) = \frac{\vec{v} \cdot \vec{w}}{\|v\| \|w\|} = \frac{1}{\sqrt{2}}$$

$\alpha = 45^\circ = \pi/4$



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Panel 22

Corollary: Two vectors $\vec{v}$ and $\vec{w}$ are perpendicular iff $\vec{v} \cdot \vec{w} = 0$

Ex: Which of the following vectors are perpendicular?

a) $\langle 1, 2, 3 \rangle$ and $\langle -1, -2, -3 \rangle$

b) $\langle 1, 2, 3 \rangle$ and $\langle -1, -3, 2 \rangle$

c) $\langle 1, 2, 3 \rangle$ and $\langle 6, -1, 1 \rangle$

d) $\langle 1, 2, 3 \rangle$ and $\langle 5, -1, 1 \rangle$

e) $\langle 1, 2, 3 \rangle$ and $\langle 0, -3, 2 \rangle$

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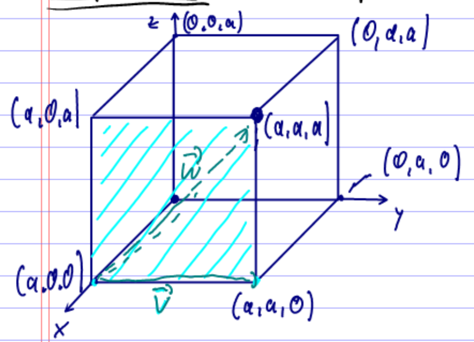
Panel 23

Application: Find angle between a diagonal and edge of a cube



First, angle of "side diag" with side is $\pi/4$

$\vec{v} = (a,a,0) - (a,0,0) = \langle 0,a,0 \rangle$

$\vec{w} = \langle 0,a,a \rangle$

$$\cos(\alpha) = \frac{v \cdot w}{\|v\| \|w\|} = \frac{\langle 0,a,0 \rangle \cdot \langle 0,a,a \rangle}{a\sqrt{2}\,a} = \frac{a^2}{\sqrt{2}a^2} = \frac{1}{\sqrt{2}}$$

3D: angle between

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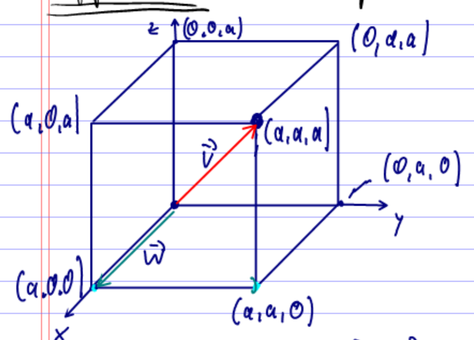
Panel 24

Application: Find angle between a diagonal and edge of a cube



$\vec{v} = \langle a,a,a \rangle$

$\vec{w} = \langle a,0,0 \rangle$

$$\cos(\alpha) = \frac{\vec{v} \cdot \vec{w}}{\|v\| \|w\|} = \frac{\langle a,a,a \rangle \cdot \langle a,0,0 \rangle}{\|\langle a,a,a \rangle\| \|\langle a,0,0 \rangle\|} = \frac{a^2+0+0}{\sqrt{a^2+a^2+a^2} \cdot \sqrt{a^2+0+0}} =$$

$$= \frac{a^2}{\sqrt{3}\,a\,a} = \frac{1}{\sqrt{3}}$$

$\alpha = 54.7°$

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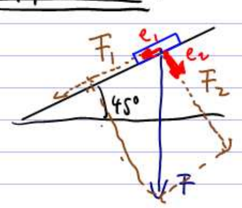
Panel 25

Application: Suppose a 10Kg block in on a 45° incline. What is the force pulling the block in the direction of the incline?



$$F = \langle 0, -10 \rangle$$

Want: $\|F_1\|$ Know: $F = F_1 + F_2$

Suppose $F = k_1 e_1 + k_2 e_2$ where e_1, e_2 are unit vectors and perpendicular

Then $F = k_1 e_1 + k_2 e_2 \quad \Big| \cdot e_1$

$e_1 F = k_1 e_1 \cdot e_1 + k_2 e_1 \cdot e_2$

$e_1 F = k_1$ is what I want!

$e_1 = \frac{1}{\sqrt{2}} \langle -1, -1 \rangle$

to find k_1 need

$F \cdot e_1 = \langle 0, -10 \rangle \cdot \frac{1}{\sqrt{2}} \langle -1, -1 \rangle$

$= \frac{10}{\sqrt{2}}$

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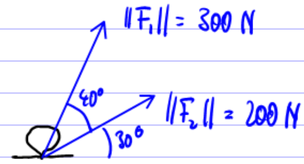
Panel 26

Application: Suppose two forces are applied to an eye bracket. Find the magnitude of the resultant force



HW

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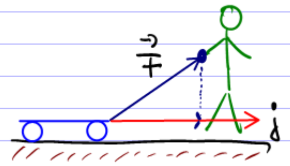
Panel 27

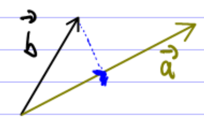
Prev. example: given 2 vectors, find resultant (no dot product)

Now: How much of a given vector $\vec{v}$ goes in direction $\vec{w}$.

Ex:



General Question:



What is projection of $\vec{b}$ onto $\vec{a}$

$\text{proj}_{\vec{a}} \vec{b} =$

Later

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