Summary 8: Derivative and Differentiability

Definition of Derivative: The derivative of a function $f(x)$ is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Alternate definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Notation:

$$f'(x) = \frac{d}{dx} f(x) = \frac{df}{dx}$$

Interpretations of Derivative:

- *Geometry*: **slope** of the tangent line
- *Physics*: (instantaneous) **velocity** if *f* is distance and *x* time
- *General*: (instantaneous) **rate of change** if *x* represents time
- *Business*: **marginal cost** if *f* is cost function and *x* the number of units produced

Power Rule:

$$\frac{d}{dx} x^p = p x^{p-1}$$

Constant Rule:

$$\frac{d}{dx} c\, g(x) = c\, g'(x)$$

Summation Rule:

$$\frac{d}{dx} \left(f(x) + g(x) \right) = f'(x) + g'(x)$$

Examples:

If $f(x) = x^3$, find $f'(3)$ using the definition:

$$f'(a) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{x^3 - 27}{x - 3} =$$

$$= \lim_{x \to 3} \frac{(x-3)(x^2+3x+9)}{(x-3)} = 27$$

If $f(x) = 2x^2 - x$, find $f'(x)$ using alt. def.:

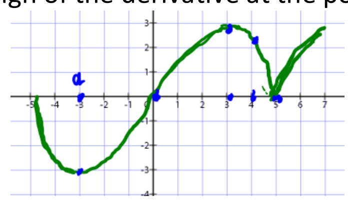
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(2(x+h)^2 - (x+h)) - (2x^2 - x)}{h} =$$

$$= \lim_{h \to 0} \frac{2x^2 + 4hx + h^2 - x - h - 2x^2 + x}{h} = \lim_{h \to 0} \frac{h(4x+h-1)}{h}$$

$$= 4x - 1$$

Find the sign of the derivative at the points:



$$f'(-3) = 0, \quad f'(0) > 0, f'(4) < 0, f'(5)\ dne$$

Find $\frac{d}{dx}$ using any appropriate rule:

$h(x) = x^{12}$

⇨ $h'(x) = 12x^{11}$

$f(x) = 5x^3$

⇨ $f'(x) = 15x^2$

$$g(x) = \frac{5}{x} - 4x^{\frac{3}{4}} + 5\sqrt[5]{x^6} + 7^2$$

$$⇨\ g'(x) = -5\,x^{-1-1} - 4\frac{3}{4}x^{\frac{3}{4}-1} + 5\frac{6}{5}x^{\frac{6}{5}-1} =$$

$$= -5x^{-2} - 3x^{-\frac{1}{4}} + 6x^{\frac{1}{5}}$$