Math 1401: Practice Exam 2

Disclaimer: This is a *practice exam* only. It is longer than the actual exam.

What is the definition of derivative f '(x)?

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

What is the Chain Rule?

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

What is a necessary condition for a function f to have a local extrema at x=c

f' is zero at x=c or undefined

What can you say about a continuous function on a closed, bounded interval and absolute extrema?

It must have abs. max and min, either at critical point or the end points.

What information about f(x) does f '(x) provide?

Incr. / Decr. / Local max/min

What information about f(x) does f ''(x) provide? Concavity / infl. point

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True/False?

If f '(x) < 0 then f is concave down F (f is decr.)

If f ''(x) = 0 then f has an inflection point at x. F Check $f(x) = x^6$

If f '(x) = 0 then f could have a maximum, or a minimum, or neither T

If f is differentiable then f must be continuous T

If f '(x) = 0 or f '(x) is undefined, then x is called critical point T

If x=c is a critical point for f, then f must have a relative extrema at x=c

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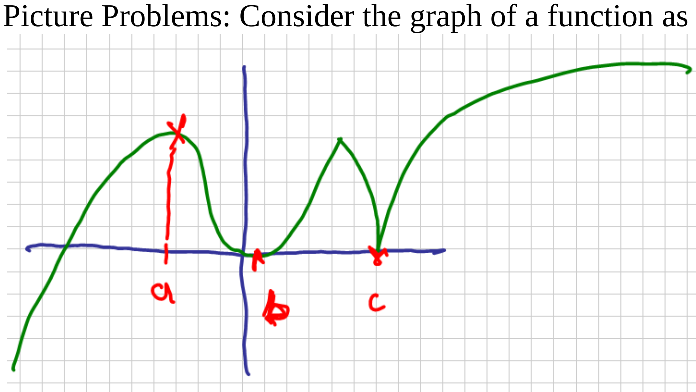
Picture Problems: Consider the graph of a function as shown:



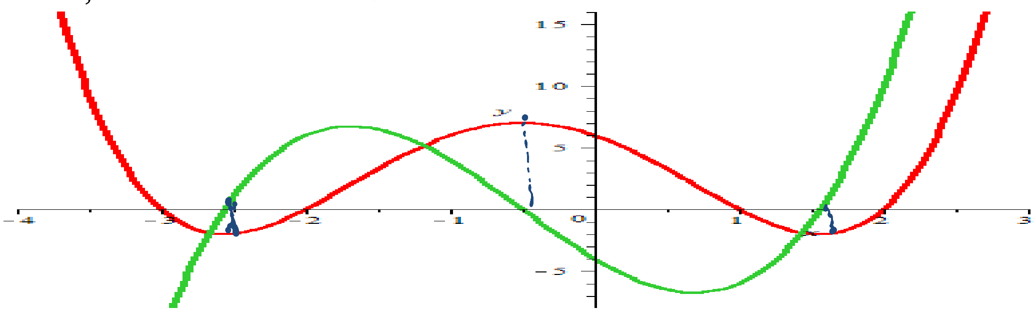
Find (sign only) if possible

$f'(a) = 0$ $\quad f''(a) < 0$

$f'(b) = 0$ $\quad f''(b) > 0$

$f'(c) = \text{undef}$ $\quad f''(c)$ undef.

In the coordinate system below you can see the graphs of *f(x)* and its derivative *f '(x)*. Who is the function, who is the derivative?

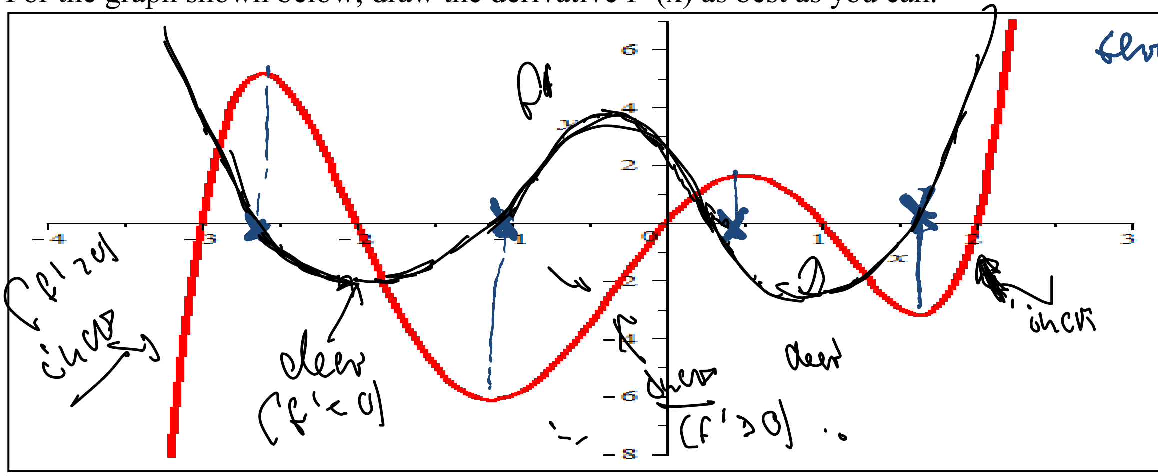


If red function has max/min, the function is red, deriv is green

For the graph shown below, draw the derivative f '(x) as best you can.

First, mark the points where f has max/min as zero, i.e. $f' = 0$

Then look for f as incr./decr. i.e. f' pos or neg



f' > 0 incr

Please find the derivative for each of the following functions (do not simplify unless it is helpful).

$f(x) = x^2(x^4 - 2x)^3$ $\quad f'(x) = 2x(x^4-2x)^3 + x^2 \cdot 3(x^4-2)^3 \cdot (4x^3 - 2)$

$f(x) = x\sin(x^2)$ $\quad f'(x) = \sin(x^2) + x\cos(x^2) \cdot 2x$

$f(x) = \dfrac{\sin(x^3)}{x^4 - 3}$ $\quad f'(x) = \dfrac{\cos(x^3) \cdot 3x^2(x^4-3) - \sin(x^3)(3x^3)}{(x^4-3)^2}$

$f(x) = \tan(x)\sqrt[3]{1-x^2}$ $\quad f'(x) = \sec^2(x)\sqrt[3]{1-x^2} + \tan(x)\frac{1}{3}(1-x^2)^{-2/3}(-2x)$

$f(x) = \pi^2 \sin\left(\sqrt{\dfrac{\pi}{6}}\right)$ $\quad f'(x) = 0$

$$f(x) = \frac{x^2\cos(1-x)}{(1-2x)^2}$$

$(1-2x)^2$

$$f(x) = x\sin(\sqrt{1-x^2})$$

$$f(x) = \sin^2(x) + \cos^2(x)$$

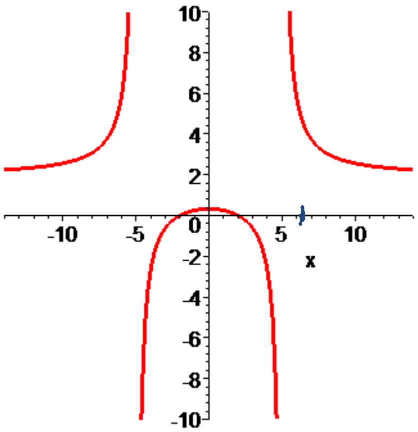
$f'(x) = 2\sin(x)$

$$f(x) = \tan(x)\text{, find } f''(x)$$

$$f(x) = \cos(x^2)\text{, find } f'''(x)$$

$2x$

For the function displayed below, find the following limits:



a) $\lim_{x\to\infty} f(x)$ $= 2$

b) $\lim_{x\to-\infty} f(x)$

c) $\lim_{x\to 5^+} f(x)$ 2

d) $\lim_{x\to -5^+} f(x)$ 2

Suppose a function y is implicitly defined as a function of x via the equation $y^3 - 5x^2 = 3x$.

a) Find the derivative of y using implicit differentiation.

$= 3y^2y'$

b) What is the equation of the tangent line at the point (1, 2).

$10 = 3$

Find the slope of the tangent line to the graph of $y^4 + 3y - 4x^3 = 5x + 1$ at the point (1, -2), assuming that the equation defines y as a function of x implicitly.

$12y' = 13$

$4y^3y' +$

$x=1, y=-2,$ $4\cdot(-8)y' + 3y' - 12\cdot 1 = 5$

$-29y' = 17 \Rightarrow y' = -\frac{17}{29}$

Find $\frac{dy}{dx}$ if $y = x^2 \sin(y)$, assuming that *y* is an implicitly defined function of *x*.

etc

$y' = 2x\sin(y) + x^2\cos(y)\, y'$

Find the following limits at infinity:

$$\lim_{x\to\infty} \frac{2x+3x^4}{4x^3-2x^2+x-1} = \infty$$

$$\lim_{x\to-\infty} \frac{x-x^5}{x^3-x^2+x-1} \sim \frac{-x^5}{x^3} = -x^2 \Rightarrow -\infty$$

$$\lim_{x\to-\infty} \frac{4x^3-2x^2+x-1}{2x-3x^4} = 0$$

$$\lim_{x\to-\infty} \frac{x^3-x^2+x-1}{x-3x^3} = -1/3$$

$$\lim_{x\to-\infty} \frac{(3x+4)(x-1)}{(2x+7)(x+2)} = 3/2$$

x^2 x^2

$$\lim_{x\to\infty} \frac{\sqrt{x^2-1}}{x} = \lim \frac{\sqrt{x^4(1-\frac{1}{x^4})}}{x} = \lim_{x\to\infty} \frac{x\sqrt{1-\frac{1}{x^4}}}{\cancel{x}} = 1$$

Find all asymptotes, horizontal and vertical, if any, for the functions

$$f(x) = \frac{3x^2+1}{9-x^2}$$

$x = \pm 3$ vertical

$y = -3$ horizontal

$$f(x) = \frac{x^5}{1+x^4}$$

no asymptotes

$$f(x) = \frac{x-3}{x^2-5x+6} = \frac{(x-3)}{(x-3)(x-2)}$$

$x = 2$ vertical asympt

$y = 0$ horizontal

If $f(x) = x^3 + x^2 - 5x - 5$, find the intervals on which f is increasing and decreasing, and find all relative extrema, if any.

$f'(x) = 3x^2 + 2x - 5 = (3x+5)(x-1) = 0 \Rightarrow x = -5/3,\ x = 1$ are critical

	inc	$-5/3$	dec	1	inc
f'	+		−		+
f	↗		↘		↗

$x = -5/3$ is rel. max

$x = 1$ is rel. min

$(-\infty, -5/3) \cup (1, \infty)$ incr.

$(-5/3, 1)$ decreasing

Determine where the function $f(x) = x^4 - 2x^2$ is increasing and decreasing and find all relative extrema, if any.

same

Find the local maxima and minima for the function $f(x) = x^{\frac{1}{3}}(8 - x)$

$$f'(x) = \frac{1}{3}x^{-2/3}(8-x) - x^{1/3} = \frac{8-x}{3x^{2/3}} - x^{1/3} = \frac{8-x}{3x^{2/3}} - x^{1/3} \cdot \frac{3x^{2/3}}{3x^{2/3}} = \frac{8-x-3x}{3x^{2/3}} = \frac{8-4x}{3x^{2/3}}$$

$\Rightarrow$ critical $x = 0, 2$ (0 because f' is undef at $x=0$)

		0		2	
f'	+		+		−
f	↗		↗		↘

$x = 2$ is local max

Find the absolute extrema (i.e. absolute maximum and absolute minimum) for the function $f(x) = 3x^4 - 6x^2$ on the interval [0, 2]

$f'(x) = 12x^3 - 12x = 12x(x^2 - 1) = 0 \Rightarrow x = 0, +1, -1$

abs. extrema:

x	$f(x)$
0	0
1	−3
−1	

← abs. min

——— not in interval

end points [——— duplicate

48 - 24 = 24 ← abs max

Find the absolute maximum and minimum of the function $f(x) = 2x^3 + 3x^2 - 36x$ on the interval [0, 4]. Do the same for $f(x) = \dfrac{x}{x^2+1}$ on [0, 3], or for $f(x) = 3x^4 + 4x^3$ on [-2, 0].

$f(x) = 2x^3 + 3x^2 - 36x \Rightarrow f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x+3)(x-2) = 0$

critical $x = -3, +2$

	x	$f(x)$	
	$\cancel{-3}$		not in interval
	2	-44	← abs. min
end pts	0	0	
	4	32	← abs. max

If $f(x) = x^3 + x^2 - 5x - 5$, determine intervals on which the graph of f is concave up and intervals on which the graph is concave down.

$f'(x) = 3x^2 + 2x - 5$

$f''(x) = 6x + 2 = 0 \Rightarrow x = -\frac{1}{3}$

		$-1/3$	
f''	$-$		$+$
f	$\cap$		$\cup$

$\Rightarrow$ Concave up: $(-1/3, \infty)$

concave down: $(-\infty, -1/3)$

If $f(x) = 12 + 2x^2 - x^4$, find all points of inflection and discuss the concavity of f. Do the same for $f(x) = x^5 - 5x^3$,

$f'(x) = 4x - 4x^3$

$f''(x) = 4 - 12x^2 = 0$

		$-1/\sqrt{3}$		$+\sqrt{1/3}$	
f''	$-$		$+$		$-$
f	$\cap$		$\cup$		$\cap$

up: $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

down: $\left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$

$x = \pm\sqrt{\frac{1}{3}}$

Find the interval where $f(x) = 1 - x^{\frac{1}{3}}$ is concave up, if any.

ok

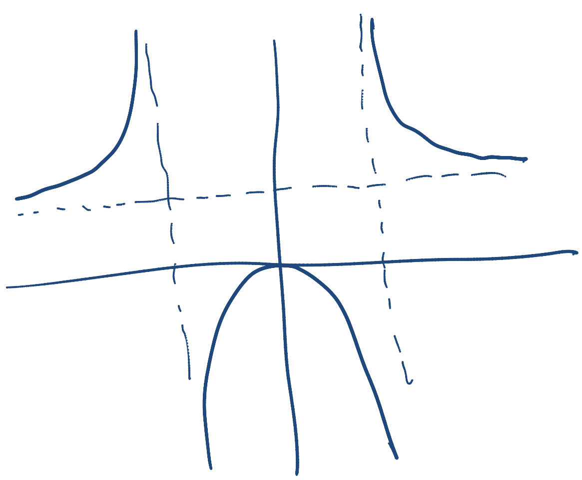
Graph the function $f(x) = \dfrac{x^2}{x^2 - 1}$. Note that $f'(x) = \dfrac{-2x}{(x^2-1)^2}$ and $f''(x) = \dfrac{2(3x^2+1)}{(x^2-1)^3}$.

horiz. asymp. $y = 1$

vert. asympt. $x = \pm 1$

critical: $x = 0, 1, -1$

poss. infl. $x = +1, -1$



		-1	0	1
f'	+	+	−	−
f''	+	−	−	+
f				

$f(0) = 0$

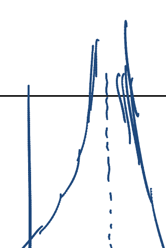
Make sure to find all asymptotes (horizontal and vertical) and clearly label any maximum, minimum, and inflection points. Then do the same for the function $f(x) = \dfrac{8-4x}{(x-1)^2}$, or ~~$f(x) = \dfrac{2x^2-8}{x^2-16}$, or $f(x) = \dfrac{x^2-1}{x^3}$~~

$f(x) = \dfrac{8-4x}{(x-1)^2}$, $f'(x) = \dfrac{4(x-3)}{(x-1)^3}$, $f''(x) = \dfrac{-8(x-4)}{(x-1)^4}$

vertical asy: $x = 1$

$f(3) = -1$

$f(4) = -\frac{8}{9}$



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horizontal asy $y=0$

critical : $x=1, x=3$

poss. infl.: $x=4, 1$

		1	3	4
f'	+	−	+	+
f''	+	+	+	−
f				

A 13 meter ladder is leaning against a wall If the top of the ladder slips down the wall at a rate of 2 m/sec, how fast will the foot be moving away from the wall when the top is 5 m above the ground.

$$x^2+y^2=13^2=169$$

$$x=x(t), \quad y=y(t)$$

Know: $\frac{dy}{dt}=-2$

$y=5 \Rightarrow x=\sqrt{169-25}=12$

$$2xx'+2yy'=0 \Rightarrow xx'=-yy' \Rightarrow x'=-y'\frac{y}{x}=2\cdot\frac{5}{12}=\frac{5}{6}$$

A liquid form of penicillin manufactured by a pharmaceutical firm is sold in bulk at a price of \$200 per unit. If the total production cost (in dollars) for x units is $C(x) = 500{,}000 + 80x + 0.003x^2$ and if the production capacity of the firm is at most 30,000 units in a specified time, how many units of penicillin must be manufactured and sold in that time to ma~~ximize~~ the p~~rofit~~ ?

minimize cost

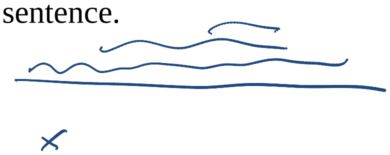
$$C(x)=500000+80x+0.003x^2, \quad x\in[0, 30000]$$

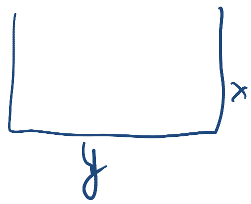
$$C'(x)=80+0.006x=0$$

$$\Rightarrow x=-13333.33 \text{ not in interval}$$

x	
0	⇐ min
~~13333.33~~	
30000	⇐ max

A farmer wants to fence in a piece of land that borders on one side on a river. She has 200m of fence available and wants to get a rectangular piece of fenced-in land. One side of the property needs no fence because of the river. Find the dimensions of the rectangle that yields maximum area. (Make sure you indicate the appropriate domain for the function you want to maximize). Please state your answer in a complete sentence.



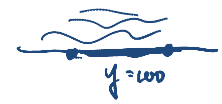


Know $2x+y=200 \Rightarrow y=200-2x$

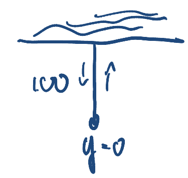
Max: $A=xy=x(200-2x)=200x-2x^2$

$x\in[0,100]$

$x=0$:



$x=100$,



$A'(x)=200-4x \Rightarrow \underline{x=50}$ is critical

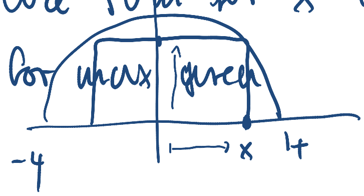
x	y
0	0
100	0

Find the dimensions of the rectangle of maximum area that can be inscribed in a semicircle of radius 16, if one vertex lies on the diameter.

$x^2+y^2=16$

Use 50 m for x and 100 meters for y $2x\cdot y$ | $\boxed{50 \quad 2500}$ max

For max square



$x^2+y^2=16 \Rightarrow y=\sqrt{16-x^2} \Rightarrow A=2x\sqrt{16-x^2}$, $x\in[0,4]$

$\left.\begin{array}{l}\Rightarrow A'=0\\ \text{Maple}\end{array}\right\} \Rightarrow x=\pm 2\sqrt{2}$

x	A
0	0
$2\sqrt{2}$	← max
4	0

An open box with a rectangular base is to be constructed from a rectangular piece of cardboard 16 inches wide and 21 inches long by cutting out a square from each corner and then bending up the sides. Find the size of the corner square which will produce a box having the largest possible volume.

$V=(16-2x)(21-2x)\cdot x$

$x\in[0,8]$

Rest use Maple