Practice Exam 3

1. Answer the following questions. Be concise:

a) What is the definition of the indefinite integral $\int f(x)\,dx$

antiderivative (function $+c$)

b) What is the *mathematical definition* of the definite integral $\int_a^b f(x)dx$?

$= \lim_{n\to\infty} f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \ldots + f(x_n)\Delta x_n$

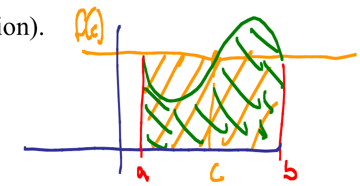
= limit of Riemann Sum

c) What is the *geometric interpretation* of the definite integral $\int_a^b f(x)dx$

area under curve $f(x)$, if f is positive

d) What is the MVT for Integrals (include a graphical interpretation).

$$\int_a^b f(x)dx = f(c)(b-a)$$



Green area = orange area

e) State the first fundamental theorem of calculus. What is that theorem good for, in your own words?

$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$, F antiderivative. Good for evaluating integrals

f) State the second fundamental theorem of calculus. What is that theorem good for, in your own words?

If $g(x) = \int_a^x f(t)dt$, f cont., then $g'(x) = f(x)$. Good for defining new functions

g) What is the difference between $\int_a^b f(x)dx$ and $\int f(x)dx$

number function

h) What is the definition of *ln(x)*, *arcsin(x)*, *arccos(x)*, and *arctan(x)*. What is the derivative of each of those functions?

All are defined as inverse functions

$\frac{d}{dx}\ln(x) = \frac{1}{x}$, $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$

i) What is l'Hopital's Rule and why is it so useful?

$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$ if $f(c) = g(c) = 0$. Useful because it applies to difficult limits

j) What is "logarithmic differentiation" and when is it helpful?

First, compute $\ln(y)$, then do implicit diff. Helpful because of the laws of logarithms

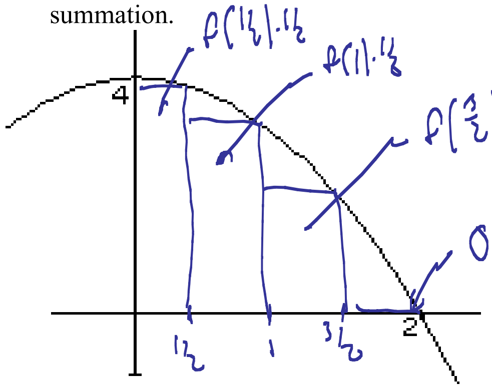
k) What is "exponential growth" and "exponential decay"?

$A = A_0 e^{kt}$, $k > 0$ $\Rightarrow$ exp growth

$A = A_0 e^{kt}$, $k < 0$ $\rightarrow$ exp. decay

2. Consider the following definite integral: $\int_0^2 4 - x^2 dx$ (The integrand is depicted below).

a) Approximate the value of that definite integral by using 4 subdivisions and **right** rectangles in the corresponding summation.



$$\Rightarrow A \approx \frac{1}{2}\left(f(\tfrac{1}{2}) + f(1) + f(\tfrac{3}{2}) + 0\right) =$$

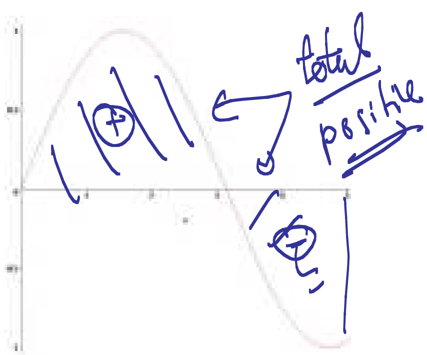
$$= \frac{1}{2}\left(4 - \frac{1}{4} + 4 - 1 + 4 - \frac{9}{4}\right) =$$

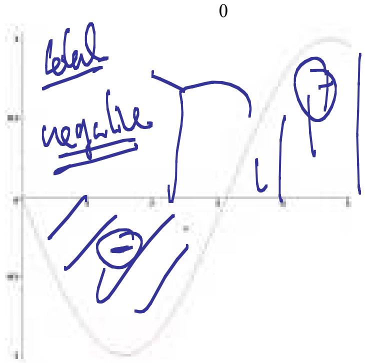
$$= \frac{1}{2}\left(11 - \frac{10}{4}\right) = \frac{34}{4}\cdot\frac{1}{2} = \frac{17}{4} = \underline{4.25}$$

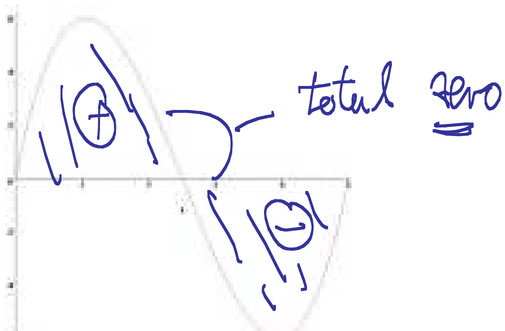
b) Find the exact value of that definite integral by using the first fundamental theorem of calculus. Compare with the answer in (a).

$$\int_0^2 4 - x^2 dx = 4x - \frac{1}{3}x^3 \Big|_0^2 = \left(8 - \frac{8}{3}\right) - 0 = \frac{16}{3} = \underline{5.33}$$

3. Below are the graphs of four functions. For each graph, decide whether $\int_0^5 f(x)dx$ is positive, negative, or zero.







4. Evaluate each of the following integrals.

$$\int 3x^2 - \frac{5}{x^3} + \sqrt[3]{x} - 2\cos(x)dx$$

$$x^3 + \frac{5}{2}x^{-2} + \frac{3}{4}x^{4/3} - 2\sin(x) + C$$

$$\int x^2 + \frac{1}{\sqrt{x}} + \pi^2 dx$$

$$\frac{1}{3}x^3 + 2x^{1/2} + \pi^2 x + C$$

$\int \frac{2}{x}dx$ $= \ln|x| + C$

$\int (x+\frac{1}{x})^2 dx$ $= \int x^2 + 2 + \frac{1}{x^2} dx = \frac{1}{3}x^3 + 2x - x^{-1} + C$

$\int \frac{5}{1+x^2}dx$ $= 5 \arctan(x) + C$

$\int 5e^x - \frac{7}{3\sqrt{1-x^2}}dx$ $= 5e^x - 7 \arcsin(x) + C$

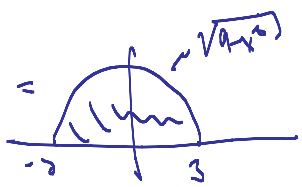
5. Evaluate each of the following definite integrals.

$\int_0^{\pi/2} \cos(x) - 3\sin(x)dx$

$= -2$

$\int_1^2 x(x-1)dx$ $= 5/6$

$\int_{-3}^{3} \sqrt{9-x^2}\,dx$ (trick question) =



$\Rightarrow$ area under ½-circle

$\frac{1}{2}\pi 3^2 = \frac{9}{2}\pi$

$\int_1^4 \frac{x-1}{\sqrt{x}}dx$ $= 8/3$

$\int_1^2 \frac{2-t^3}{t^2}dt$ $= -1/2$

$\int_\pi^\pi \cos(4x^2)dx$ (trick question) $\int_\pi^\pi \text{anything}\, dx = 0$

6. At SLAC (Stanford Linear Accelerator) the initial position of a particle was recorded at time $t = 0$ to be 10m. Several detectors were used to record the speed of the particle, and it could be determined that the velocity function of the particle was given by $v(t) = 4t^2 + 3t$. What is the distance function of the particle, and where is the particle after 5 seconds?

$v(t) = 4t^2 + 3t \rightarrow s(t) = \frac{4}{3}t^3 + \frac{3}{2}t^2 + C.$ $s(0) = 10 \rightarrow C = 10$

7. Suppose that gasoline is increasing in price according to the equation $p(t) = 1.2 + 0.04t$ where *p* is the dollar price per gallon, and *t* is the time in years, with $t = 0$ representing 1990. If an automobile is driven 15,000 miles a year, and gets M miles per gallon, the annual fuel cost is $C = \frac{15000}{M}\int_t^{t+1} p(t)dt$. Estimate the annual fuel cost (a) for the year 2000 and (b) for the year 2005.

a) $\frac{15000}{M}\int_{10}^{11} 1.2 + 0.04t \, dt$

b) $\frac{15000}{M}\int_{15}^{16} 1.2 + 0.04t \, dt$

8. Define a function $S(x) = \int_0^x \cos^2(t^2)dt$. Then find S(0), S'(x), and S''(x). Where is the function increasing and decreasing?

$S(0) = \int_0^0 - \, dt = 0$

$S'(x) = \cos^2(x^2) \Rightarrow S''(x) = 2\cos(x^2)\cdot(-\sin(x^2))\cdot 2x$

Also, if $G(x) = \int_x^{x^2} \sin(t^2)dt$ then find G(1) and G'(x).

$G(1) = \int_1^1 - \, dt = 0$ $\quad G'(x) = $ (did not do this part)

Finally, define $H(x) = \int_0^x 5e^t - 3t^2 dt$. Find H(x) using the 1st Fund. Theorem of Calculus, then compute the derivative of that function. Compare that with $\frac{d}{dx}H(x)$ as worked out with the 2nd Fund. Theorem of Calc.

$H(x) = \int_0^x 5e^t - 3t^2 dt = 5e^t - t^3 \Big|_0^x = (5e^x - x^3) - (5) \Rightarrow$

$H'(x) = \frac{d}{dx}(5e^x - x^3 - 5) = 5x - 3x^2$ matches 2nd Fund. Theorem

9. Find the derivatives of the following functions. You might use logarithmic differentiation if that simplifies your task:

$f(x) = 2e^{5x^2} + 7\ln(x^2+3)$

$f'(x) = 2e^{5x^2} \cdot 10x + 7/(x^2+3) \cdot 2x$

$f(x) = \ln\left(\frac{\sqrt{x-1}}{(x-1)^2}\right) = \frac{1}{2}\ln(x-1) - 2\ln(x-1) = -\frac{3}{2}\ln(x-1)$

$\Rightarrow f'(x) = -\frac{3}{2}\frac{1}{x-1}$

$f(x) = (x-1)^2\ln\left(\sqrt{x^2-1}\right) = (x-1)^2 \cdot \frac{1}{2}\ln(x^2-1)$

$f'(x) = 2(x-1)\cdot 1 \cdot \ln\sqrt{x^2-1} + (x-1)^2 \cdot \cancel{\frac{1}{2}} \frac{1}{x^2-1} \cdot \cancel{2}x$

$y = f(x) = \frac{(x-1)^2}{(x+1)^3}(x+2)^4$

$\ln(y) = 2\ln(x-1) + 4\ln(x+2) - 3\ln(x+1)$

$\Rightarrow \frac{1}{y}\cdot y' = \frac{2}{x-1} + 4/(x+2) - \frac{3}{x+1} \quad \Rightarrow y' = \left(\frac{2}{x-1} + \frac{4}{x+2} - \frac{3}{x+1}\right)\cdot y$

$g(x) = \frac{\sqrt{x^2-1}}{x^5(x-4)^4}$

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10. Simplify $\log_2(\frac{x^3y}{z^4})$ and $2\ln(4) - \ln(2)$

$\log_2\left(\frac{x^3y}{z^4}\right) = 3\log_2(x) + \log_2(y) - 4\log_2(z)$, $2\ln(4) - \ln(2) = \ln\left(\frac{4^2}{2}\right) = \ln(8)$

11. The half-life or radium-226 is 1590 years. A sample of radium-226 has a mass of 100 mg. Find a formula about how much of the substance remains after t years. Find the mass after 1000 years. Also, find out how long it takes until the original mass of 100mg is reduced to 80 mg.

$A = A_0 e^{kt} \Rightarrow A(1590) = \frac{1}{2}A_0 = A_0 e^{k\cdot 1590} \Rightarrow \frac{1}{2} = e^{k\cdot 1590} \Rightarrow \underline{\ln(\frac{1}{2})\cdot\frac{1}{1590} = k}$

$\Rightarrow A(t) = 100\cdot e^{-\frac{\ln(2)}{1590}\cdot t} \quad \Rightarrow A(1000) = 100e^{-\frac{\ln(2)}{1590}\cdot 1000} = \underline{64.66}$

$A(t) = 80 = 100e^{-\frac{\ln(2)}{1590}t} \Rightarrow 0.8 = e^{-\frac{\ln(2)}{1590}t} \Rightarrow -\ln(0.8)\cdot\frac{1590}{\ln(2)} = \underline{t = 511}$

12. Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960.Assuming exponential growth, what would the population of the world be in 2020?

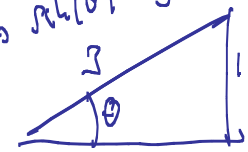
13. Evaluate $\sin^{-1}(\frac{1}{2})$ and $\tan\left(\arcsin\left(\frac{1}{3}\right)\right)$

$\sin^{-1}(\frac{1}{2}) = \theta$

$\Rightarrow \frac{1}{2} = \sin(\theta) \Rightarrow \theta = \pi/6$

$\tan(\arcsin(1/3))$

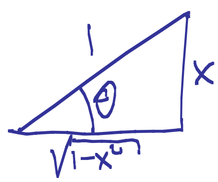
$\theta = \arcsin(1/3) \Rightarrow \sin(\theta) = 1/3$



$\Rightarrow \tan(\theta) = 1/\sqrt{8}$

14. Show that $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$

$\theta = \sin^{-1}(x) \Rightarrow \sin(\theta) = x = \frac{x}{1}$



$\Rightarrow \cos(\theta) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

15. Find the derivatives of $f(x) = x^2\sin^{-1}(3x)$ and $g(x) = \frac{\arctan(3x)}{\arccos(2x)}$ and $h(x) = x\sqrt{\tan^{-1}(x^2)}$

$f'(x) = 2x\sin^{-1}(3x) + x^2\frac{1}{\sqrt{1-(3x)^2}}\cdot 3$

$h'(x) = \sqrt{\tan^{-1}(x^2)} + x\frac{1}{2}\left(\tan^{-1}(x^2)\right)^{-1/2}\cdot\frac{1}{1+x^4}\cdot 2x$

$g'(x) = \frac{\frac{3}{1+(3x)^2}\cdot\arccos(2x) - \arctan(3x)\cdot\frac{(-2)}{\sqrt{1-(2x)^2}}}{(\arccos(2x))^2}$

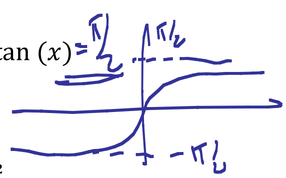
16. Prove that $\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$ for all x. *Hint: Find the derivative and interpret your answer*

Let $F(x) = \sin^{-1}(x) + \cos^{-1}(x) \rightarrow F'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$ for all x.

$\Rightarrow F$ is constant. Since $F(0) = 0 + \pi/2$, that constant is $\pi/2$.

17. Find the following limits. You might want to use l'Hospital's rule (but not all limits require that and for some it might not be appropriate)

$\lim_{x\to\infty}\arctan(x) = \pi/2$



$\lim_{x\to\infty} e^{-x^2}$

$= 0$

$\lim_{x\to 3}\frac{x^2-9}{x+3} = \frac{0}{6} = 0$

$\lim_{x\to 1}\frac{\ln(x)}{x-1} \overset{l'H}{=} \lim_{x\to 1}\frac{1/x}{1} = 1$

$\lim_{x\to 3}\frac{x^2-9}{x-3} \overset{l'H}{=} \lim_{x\to 3}\frac{2x}{1} = 6$

$\lim_{x\to\infty}\frac{x^2}{e^x} \overset{l'H}{=} \lim_{x\to\infty}\frac{2x}{e^x} \overset{l'H}{=} \lim_{x\to\infty}\frac{2}{e^x} = 0$

$\lim_{x\to 0}\frac{\tan(x)-x}{x^3} \overset{(l'H)}{=} \lim_{x\to 0}\frac{\sec^2(x)-1}{3x^2} = \lim_{x\to 0}\frac{\tan^2(x)}{3x^2} \overset{(l'H)}{=} \lim_{x\to 0}\frac{2\tan(x)\cdot\sec^2(x)}{6x} =$

trig identity

$= \lim_{x\to 0}\frac{2\sec^4(x)+2\tan(x)\,2\sec^2(x)\tan(x)}{6} = \frac{2}{6}$

$= 1/3$

$\lim_{x\to 0}\frac{\cos(x^2)-1}{5x^4} = \lim_{x\to 0}\frac{-\sin(x^2)\cdot 2x}{20x^3} \overset{(l'H)}{=} \lim_{x\to 0}\frac{-\sin(x^2)}{10x^2} \overset{(l'H)}{=}$

$= \lim_{x\to 0}\frac{-\cos(x^2)\cancel{2x}}{\cancel{20x}} = -1/10$

$\lim_{x\to\pi}\frac{\sin(x)}{1-\cos(x)} = \frac{0}{2} = 0$

Cuo l'Hospital!!

$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+}\frac{\ln(x)}{1/x} \overset{(l'H)}{=} \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0$

$\lim_{x\to\frac{\pi}{2}^+} \sec(x)-\tan(x)$

Use Wolfram Alpha!