Panel 1

Last time:

Intermediate Value Theorem

- Limits involving infinity
  - vertical asymptote
  - horizontal asymptote

Limits at infinity of rational functions

Panel 2

Most common IVT application

Suppose \( f \) is continuous on \((a, b)\) and

\[ f(a) < 0 \quad \text{and} \quad f(b) > 0 \quad \text{or vice versa} \]

Then \( \exists \ c \in (a, b) \) such that \( f(c) = 0 \).

\( f(c) \) lies at least on \( c \) in \((a, b)\)

\( f(c) \) between \( m \) and \( M \)

\[ \exists x: \cos(x) = x \quad \text{on} \quad (0, 1) \]

**IVT:**

\[ \sin(x) = x \quad \text{on} \quad (0, 1) \]

Then \( f(0) = \cos(0) - 0 = 1 > 0 \)

\( f(1) = \cos(1) - 1 < 0 \)

also \( f \) in continuous \( \Rightarrow \) by IVT there is \( c \) with \( f(c) = 0 \)
Panel 3

\[ \cos(x) = x \]

Panel 4

Limits Graphically

\[ \lim_{x \to 0^+} f(x) = +\infty \]
\[ \lim_{x \to 0^-} f(x) = -\infty \]
\[ \lim_{x \to 1^+} f(x) = 1 \]
\[ \lim_{x \to 1^-} f(x) = -1 \]
\[ \lim_{x \to -\infty} f(x) = -1 \]
\[ \lim_{x \to +\infty} f(x) = 1 \]

4. \( \lim_{x \to 0^-} f(x) = 0 \)
5. \( \lim_{x \to 0^+} f(x) = +\infty \)
6. \( \lim_{x \to \pm \infty} f(x) = 1 \)
7. \( \lim_{x \to 0} f(x) = d.n.e. \)
Panel 5

**Horizontal Asymptotes of Rational Functions**

\[
\lim_{x \to \pm \infty} \frac{p(x)}{q(x)}, \quad \text{where } p(x) \text{ and } q(x) \text{ are polynomials}
\]

When in doubt, factor and (highest power)

\[
\lim_{x \to \pm \infty} \frac{p(x)}{q(x)} = \begin{cases} 
\# & \text{if } \deg(p) = \deg(q) \\
0 & \text{if } \deg(p) < \deg(q) \\
\pm \infty & \text{if } \deg(p) > \deg(q)
\end{cases}
\]

Panel 6

Limits involving infinity are based on:

\[
\lim_{x \to \infty} \frac{1}{x} = 0 \quad \lim_{x \to \infty} \frac{1}{x} = \infty \\
\lim_{x \to -\infty} \frac{1}{x} = 0 \quad \lim_{x \to -\infty} \frac{1}{x} = -\infty
\]

\[
\lim_{x \to 0} \frac{1}{x} = \text{d.n.e.}
\]

\[
\frac{8}{0} \quad \text{or} \quad \frac{0}{0} \quad \text{require more work!}
\]
Panel 7

**Horizontal Asymptotes of Rational Functions**

\[ \lim_{x \to \pm \infty} \frac{p(x)}{q(x)}, \text{ where } p(x) \text{ and } q(x) \text{ are polynomials} \]

Panel 8

**Gravitational Force**

\[ F(r) = \begin{cases} \frac{GM}{r^2}, & r < R \\ \frac{GM}{r^2}, & r \geq R \end{cases} \]

Question: is \( F \) continuous?

\[ \lim_{r \to R^-} F(r) = \lim_{r \to R^+} \frac{GM}{r^2} = \frac{GM}{R^2} \]

\[ \lim_{r \to R^-} F(r) = \lim_{r \to R^+} \frac{GM}{R^2} = \frac{GM}{R^2} \]

\[ \frac{GM}{R^2} = F(R) \]
Panel 9

\[ x^3 + x - 3 = 0 \] has a solution on \((1, 2)\)

1. \(x^3 + x - 3\) is cont. on \((1, 2)\) \(\Rightarrow\) Thus, there is a \(c\) such that \(f(c) = 0\).
2. \(f(1) = -1 < 0\)
   \[ f(2) = 15 > 0 \]

Prove that there is a number such that \(e^2 = 2\)
\[ f(x) = x^2 - 2 = 0 \]
\[ f(1) < 0 \Rightarrow \text{there is a number between} \]
\[ f(2) > 0 \]

in fact, \(x = \sqrt{2} \approx 1.414\ldots\)

Panel 10

\[ \lim_{x \to \pi^-} \tan(x) = \lim_{x \to \pi^-} \frac{\sin(x)}{\cos(x)} = 0 \]
\[ \lim_{x \to \pi^+} \cos(x) = \lim_{x \to \pi^+} \frac{\cos(x)}{\sin(x)} = -1 \pm 0 \]

\[ \lim_{x \to \pi^-} \cos(x) = \lim_{x \to \pi^-} \frac{\cos(x)}{\sin(x)} = -1 \pm 0 \]

Panel 11

\[
\begin{align*}
\lim_{x \to \infty} \left( x - \sqrt{x^2 - x} \right) &= \infty - \infty \\
&= \lim_{x \to \infty} \frac{x - \sqrt{x^2 - x}}{x + \sqrt{x^2}} \\
&= \lim_{x \to \infty} \frac{x^2 - x}{x^2 + x \sqrt{x^2}} \\
&= \lim_{x \to \infty} x \cdot \frac{1 - \frac{x}{x^2}}{1 + \frac{x}{x} \sqrt{1}} \\
&= \lim_{x \to \infty} x \cdot 1 = \infty
\end{align*}
\]

\[
\begin{align*}
\lim_{x \to \infty} \frac{x^3 - 2x + 3}{5 - 2x^3} &= \frac{\infty}{-\infty} \\
&= -\frac{1}{2} \quad \checkmark
\end{align*}
\]

∞ - ∞ ?
∞ + ∞ = ∞
-∞ + ∞ ?
-∞ - ∞ = -∞

Panel 12

\[
\lim_{x \to \infty} \frac{x^3 - 5x}{2 - x^2} = 0
\]

\[
\begin{align*}
\lim_{x \to \infty} \frac{x^3 - 5x}{2 - x^2} &= \lim_{x \to \infty} \frac{x^3 \left(1 - \frac{5}{x^3}\right)}{x^2 \left(1 - \frac{2}{x^2}\right)} \\
&= \lim_{x \to \infty} x \cdot (1 - 1) = 0
\end{align*}
\]

\[
\begin{align*}
\lim_{x \to -\infty} \frac{x^3 - 5x}{2 - x^2} &= \lim_{x \to -\infty} \frac{x^3 \left(1 - \frac{5}{x^3}\right)}{x^2 \left(1 - \frac{2}{x^2}\right)} \\
&= \lim_{x \to -\infty} x^2 (-1) = -\infty
\end{align*}
\]
Quiz #2

1. Show that \( x^4 - x^2 - 1 = 0 \) has at least one solution in the interval \((0,2)\).

2. Consider the function shown and find the given limits:

   \[ \lim_{x \to 2^+} f(x) \]

   \[ \lim_{x \to 2^-} f(x) \]

   \[ \lim_{x \to -\infty} f(x) \]

Panel 14

3. Find the following limits:

   a) \( \lim_{x \to 2^-} \frac{3x + 1}{x - 2} \)

   b) \( \lim_{x \to \infty} \frac{3 - x^2}{1 + x + 2x^2} \)

   c) \( \lim_{x \to -\infty} \frac{5x^2 + 4x - 9}{6x^3 + 3x^2 - 7x} \)
Panel 15

Done with limits. Next we'll do more limits!

Tangent problem: Find tangent line to $y = x^2$

at $x = 1$

Tangent line is a line that just "touched" the graph at that point.

Ex: Tangent line at $x = 0$ has equation $y = 0$

Tangent line at $x = 1$ has positive slope.

Panel 16

Chapter 2: Differentiation

Let $f(x) = 2x^2 + 1$, find line $\frac{f(x) - f(1)}{x-1}$.
Panel 17

\[ f(x) = x^2 \]

Slope of this line is

\[ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

**Def:** Slope of tangent line to a function \( f \) is:

\[ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

Panel 18

In our case: \( f(x) = x^2 \), at \( x = 1 \)

Slope is \( \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^2 - 1}{h} \)

\[ = \lim_{h \to 0} \frac{1+2h+h^2 - 1}{h} = \lim_{h \to 0} \frac{h(2+h)}{h} = 2 \]

Slope is \( m = 2 \) \( \frac{\text{known}}{\text{known}} \) \( f(x) = mx + b = 2x + b \)

Known line goes through \( x = 1, y = 1 \)

\[ \Rightarrow f(1) = 1 = 2(1) + b \Rightarrow b = -1 \]
Panel 19

\[ f(x) = 2x - 1 \]

The limit \( \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \) has a special name: slope of tangent derivative of a function \( f(x) \)

Panel 20

Find the derivative of \( f(x) = x^2 - 8x + 9 \)

\[
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - 8(x+h) + 9 - (x^2 - 8x + 9)}{h}
\]

\[
= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 8x - 8h + 9 - x^2 + 8x - 9}{h}
\]

\[
= \lim_{h \to 0} \frac{2xh + h^2 - 8h}{h} = \lim_{h \to 0} \frac{h(2x + h - 8)}{h} = 2x - 8
\]

Goal: Find quick ways to compute the Derivative.
Panel 21

Graphically, \( f' \) is "easy" to see!

Panel 22

How to compute derivative quickly algebraically:

Next time!

will post HW

Ex 1 on Monday