Panel 1

Last time:

Limits — computationally, graphically, definition

Squeezing Theorem

Continuity — limits, geometrically, piecewise defined

Theorems involving continuity

Quiz on Monday!

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Panel 2

$$\lim_{x\to 0} \frac{\sin(3x)}{x} = \lim_{x\to 0} \frac{\cancel{3\sin(x)}}{x} = 3$$

Know: $\lim_{t\to 0} \frac{\sin(t)}{t} = \lim_{t\to 0} \frac{\sin(\square)}{(\square)} = 1$

$$= \lim_{x\to 0} \frac{\sin(3x)}{x} = \lim_{x\to 0} \frac{3\cdot \sin(3x)}{3x} = 3 \lim_{x\to 0} \frac{\sin(3x)}{3x} = 3\cdot 1$$

$$\frac{\sin(3x)}{x} = \frac{3}{3}\cdot \frac{\sin(3x)}{x} = \frac{3\sin(3x)}{3x}$$

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Panel 3

Continuity Theorem

If f and g are continuous, so are $f+g$, $f-g$, $f\cdot g$, and f/g (if $g \neq 0$), and $f(g(x))$ if defined.

Ex: Where is $f(x) = \dfrac{x^2+9}{5-3x}$ continuous ?

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Panel 4

$$\lim_{x\to 0} \frac{\sin(4x)}{\sin(6x)} = \lim_{x\to 0} \frac{\frac{1}{x}\,\sin(4x)}{\frac{1}{x}\,\sin(6x)} = \lim_{x\to 0} \frac{4\,\frac{\sin(4x)}{4x}}{6\,\frac{\sin(6x)}{6x}}$$

$$\frac{\sin(t)}{(t)}$$

$$= \lim_{x\to 0} \frac{4}{6} \cdot \frac{\frac{\sin(4x)}{4x}}{\frac{\sin(6x)}{6x}} \quad \begin{matrix} 1 \\ 1 \end{matrix} = \frac{4}{6} \cdot 1 = \underline{\frac{2}{3}}$$

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Panel 5

$$\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)=\lim_{t\to 0}\frac{t^2+t}{t\cdot(t^2+t)}-\frac{t}{t(t^2+t)}=$$

$$=\lim_{t\to 0}\frac{t^2+t-t}{t(t^2+t)}=$$

$$=\lim_{t\to 0}\frac{t^{\cancel{2}}}{\cancel{t}(t^2+t)}=\lim\frac{t}{t^2+t}=$$

$$=\lim_{t\to 0}\frac{\cancel{t}}{\cancel{t}(t+1)}=\lim_{t\to 0}\frac{1}{t+1}=1$$

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Panel 6

$$\lim_{h\to 0}\frac{(3+h)^{-1}-3^{-1}}{h}=\lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h}=$$

$$=\lim_{h\to 0}\frac{\frac{(3)-(3+h)}{3(3+h)}}{h}=$$

$$=\lim_{h\to 0}\frac{\frac{-h}{3(3+h)}}{h}=$$

$$=\lim_{h\to 0}\frac{1}{\cancel{h}}\cdot\frac{-\cancel{h}}{3(3+h)}=-1/9$$

6

Panel 7

Intermediate Value Theorem

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Panel 8

$$\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{|x|} \right) :$$

x is positive

HW $\Rightarrow$ $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{|x|} \right) = \lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{x} \right) = 0$

$\lim_{x \to 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right) = \lim_{x \to 0^-} \left(\frac{1}{x} + \frac{1}{x} \right) = \lim_{x \to 0^-} \frac{2}{x} =$ inf. now due.

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Panel 9

$$g(x) = \begin{cases} x^2 - c^2, & x < 4 \\ cx + 20, & x \geq 4 \end{cases}$$

find c s.t. g is continuous on $(-\infty, \infty)$

$$\left[f(c) = \lim_{x \to c} f(x) \right]$$

$f(4) = 4c + 20$ ✓

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} x^2 - c^2 = 16 - c^2$$

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} cx + 20 = 4c + 20$$

$$16 - c^2 = 4c + 20$$

$$0 = c^2 + 4c + 4 = (c+2)^2 \Rightarrow \underline{\underline{c = -2}}$$

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Panel 10

Continuity Theorem

If f and g are continuous, so are $f+g$, $f-g$, $f \cdot g$, and f/g (if $g \neq 0$), and $f(g(x))$ if defined.

Ex: Where is $f(x) = \dfrac{x^2 + 9}{5 - 3x}$ continuous?

Must exclude $5 - 3x = 0 \Rightarrow$ exclude $x = 5/3$

continuous for all x except $\underline{\underline{5/3}}$

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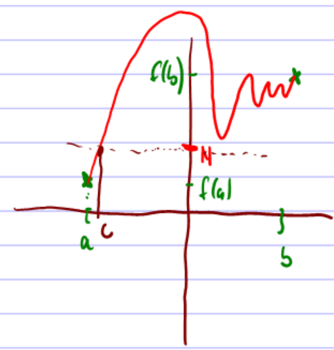
Panel 11

Intermediate Value Theorem

Suppose f is continuous on $[a,b]$ and N is a number between $f(a)$ and $f(b)$. Then there is a number c in interval (a,b) with $f(c)=N$.



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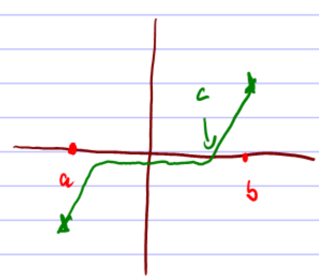
Panel 12

Most common IVT application

f continuous, $f(a)<0$, $f(b)>0$ then there is a c such that $f(c)=0$



Ex: Show that the expression

$$4x^3-6x^2+3x-2=0$$

has a solution between 1 and 2

$f(1)=4-6+3-2=-1<0$

$f(2)=32-24+6-2=12>0$

$\Rightarrow$ there is $f(c)=0$

How to find that number?

Is it 1.5? $f(1.5)>0$

numeric way of finding two

bisection method

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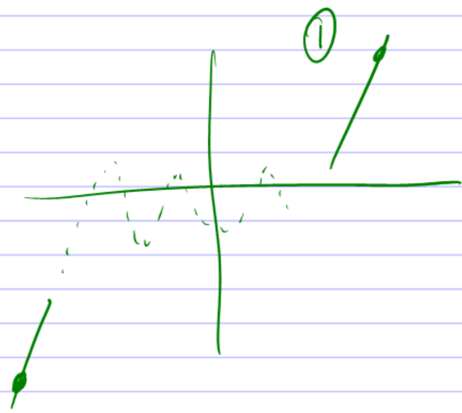
Panel 13

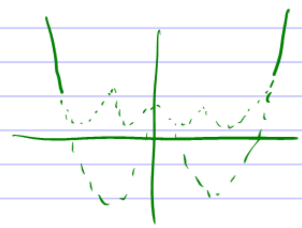
Which of these MUST have an answer?

① $x^9 - x^8 + x^7 - 5x^3 + 2x^2 - 9 = 0$

② $x^{16} - x^4 + x^3 - x + 900 = 0$



① must have a solution



② may have a solution

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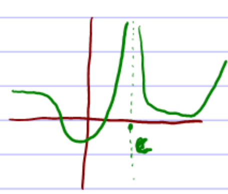
Panel 14

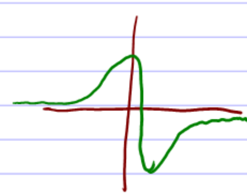
Limits involving Infinity

2 types: ⓐ $\lim\limits_{x \to \pm\infty} f(x)$

ⓑ $\lim\limits_{x \to a} f(x) = \pm\infty$ start



type ⓑ



type ⓐ

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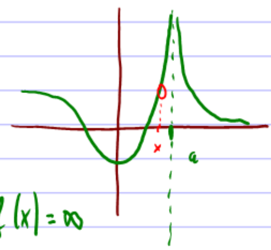
Panel 15

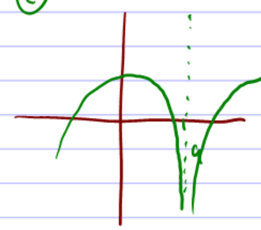
(B) $\lim_{x \to a} f(x) = \pm\infty$

①



$\lim_{x \to a} f(x) = \infty$

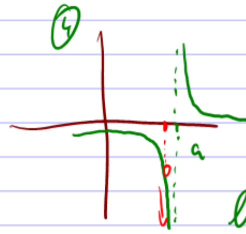
②



$\lim_{x \to a} f(x) = -\infty$

③

④



$\lim_{x \to a^-} f(x) = +\infty$

$\lim_{x \to a^+} f(x) = -\infty$

$\Big\}$ $\lim_{x \to a} f(x)$ undef. $<$

$\lim_{x \to a^-} f(x) = -\infty$

$\lim_{x \to a^+} f(x) = \infty$

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Panel 16

Ex:

① $\lim_{x \to 0} \frac{1}{x} =$ undef $\begin{cases} \lim_{x \to 0^+} \frac{1}{x} = \infty \\ \lim_{x \to 0^-} \frac{1}{x} = -\infty \end{cases}$

② $\lim_{x \to 0} \frac{1}{x^2} = +\infty$

③ $\lim_{x \to 0} \frac{1}{x^5}$ undefined

④ $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$ as before!

⑤ $\lim_{x \to 1} \frac{x^2 + 1}{x - 1} =$ undef. $\begin{cases} \lim_{x \to 1^+} \frac{x^2+1}{x-1} = +\infty \\ \lim_{x \to 1^-} \frac{x^2+1}{x-1} = -\infty \end{cases}$

If $\lim_{x \to a} f(x) = \infty$

or $\lim_{x \to a^{+/-}} f(x) = \pm\infty$

then $x = a$ is vertical asymptote

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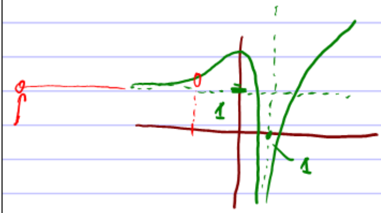
Panel 17

④ $\lim_{x \to \pm\infty} f(x)$



① $\lim_{x \to -\infty} f(x) = 1$

② $\lim_{x \to 1} f(x) = -\infty$

If $\lim_{x \to \pm\infty} f(x) = L$ then

$y = L$ is called

horizontal asymptote

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Panel 18

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Panel 19

Horizontal Asymptotes of Rational Functions

$\lim_{x\to\infty} \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials

[Trick: factor highest powers]

Ex: $\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2+4x+1} = \lim_{x\to\infty} \frac{\cancel{x^2}(3 - \frac{1}{x} - \frac{2}{x^2})}{\cancel{x^2}(5 + \frac{4}{x} + \frac{1}{x^2})} = \frac{3}{5}$

$\lim_{x\to-\infty} \frac{2x^2-1}{5x+3x^3} = \lim_{x\to-\infty} \frac{\cancel{x^2}(2-\frac{1}{x^2})}{\cancel{x^3}(\frac{5}{x^2}+3)} = \lim_{x\to-\infty} \frac{1}{x}\cdot\frac{2}{3} = 0$

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Panel 20

Ex: $\lim_{x\to\infty} \frac{x^4 - 3x^2}{5x^4+9x^3+7} = \lim_{x\to\infty} \frac{\cancel{x^4}(1 - \frac{3}{x^2})}{\cancel{x^4}(5 + \frac{9}{x} + \frac{7}{x^4})} = \frac{1}{5}$

$\lim_{x\to-\infty} \frac{x+9}{x^3+9x+9} = \lim_{x\to-\infty} \frac{1\,\cancel{x}(1+\frac{9}{x})}{x^2\cancel{x^3}(1+\frac{9}{x^2}+\frac{9}{x^3})} = \lim_{x\to-\infty} \frac{1}{x^2}\cdot\frac{1}{1} = 0$

$\lim_{x\to\infty} \frac{x^3+8}{x-2} = \frac{x^3}{x}\ldots = x^2 \ldots = \infty$

$\lim_{x\to\infty} \frac{5-x-3x^2}{9+2x^2} = \frac{\cancel{x^2}}{\cancel{x^2}}\left(\frac{-3}{2}\right) = -\frac{3}{2}$

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Panel 21

Note:

$$\lim_{x \to \pm\infty} \frac{p(x)}{q(x)} = \begin{cases} \# & \text{if degree of } p = \text{degree of } q \\ \pm\infty & \text{if degree of } p < \text{degree of } q \\ 0 & \text{if degree of } p > \text{degree of } q \end{cases}$$

$$\lim_{x \to \infty} \frac{x^3 + 9x}{4x + x^5} = 0$$

$$\lim_{x \to \infty} \frac{4x^2 - 9x^4}{5x + 3x^3} \sim \lim_{x \to \infty} \frac{-9x^4}{3x^3} \cdots \lim_{x \to \infty} -\frac{9}{3}x = -\infty$$

plenty to practice on HW → QZ on Monday

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