

# TUTTE SETS IN GRAPHS II: THE COMPLEXITY OF FINDING MAXIMUM TUTTE SETS \*

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June 11, 2005

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\*Part of this research was done while the first two authors were visiting the Università di Roma. The hospitality and financial support are gratefully acknowledged.

## Abstract

A well-known formula of Tutte and Berge expresses the size of a maximum matching in a graph  $G$  in terms of what is usually called the deficiency. A subset  $X$  of  $V(G)$  for which this deficiency is attained is called a Tutte set of  $G$ . While much is known about maximum matchings, less is known about the structure of Tutte sets. We explored the structural aspects of Tutte sets in another paper. Here we consider the algorithmic complexity of finding Tutte sets in a graph. We first give two polynomial algorithms for finding a maximal Tutte set. We then consider the complexity of finding a maximum Tutte set, and show it is NP-hard for general graphs, as well as for several interesting restricted classes such as planar graphs. By contrast, we show we can find maximum Tutte sets in polynomial time for graphs of level 0 or 1, elementary graphs, and 1-tough graphs.

*Keywords:* (perfect) matching, Tutte set, extreme set, deficiency,  $D$ -graph

*AMS Subject Classifications (2004):* 05C70, 05C75

## 1 Introduction

In this paper we consider only simple graphs. Our terminology will be standard. Good references for any undefined terms are [5] and [9].

Given a graph  $G$ , let  $\omega(G)$  (resp.,  $\omega_0(G), \omega_e(G)$ ) denote the number of components (resp., odd, even components) of  $G$ . An important result in matching theory is due to Tutte [8].

**Theorem 1.1** (*Tutte's Theorem*) *A graph  $G$  has a perfect matching if and only if  $\omega_0(G - X) \leq |X|$  for all  $X \subseteq V(G)$ .*

In 1958, Berge [3] extended Tutte's Theorem to give the exact size of a maximum matching in a graph  $G$ . Define the **deficiency** of  $G$ , denoted  $\text{def}(G)$ , by  $\max_{X \subseteq V(G)} \{\omega_0(G - X) - |X|\}$ , where the maximum is taken over all proper subsets of  $V(G)$ .

It can be shown that  $\text{def}(G)$  is the number of vertices unmatched by a maximum matching in  $G$ , and thus we have the following.

**Theorem 1.2** (*Tutte-Berge Formula*) *The maximum size of a matching in a graph  $G$  is  $\frac{|V(G)| - \text{def}(G)}{2}$ .*

Motivated by the above formula, we define a **Tutte set** in  $G$  to be a subset  $X \subseteq V(G)$  such that  $\omega_0(G - X) - |X| = \text{def}(G)$ . These sets were referred to as barriers in [7].

In [1], we studied the structure of maximal Tutte sets in graphs. In this note we consider the algorithmic complexity of finding maximal and maximum Tutte sets in graphs.

We begin with some necessary definitions and theorems from [1].

Let  $G$  be a graph. The Edmonds-Gallai decomposition of  $G$  is the partition  $D_G \cup A_G \cup C_G$  of  $V(G)$  given by

- $D_G = \{v \in V(G) \mid \text{some maximum matching in } G \text{ fails to match } v\}$
- $A_G = \{u \in V(G) - D_G \mid u \text{ is adjacent to a vertex in } D_G\}$
- $C_G = V(G) - D_G - A_G$ .

In what follows, we omit the subscript  $G$ , if understood.

In particular, if  $G$  contains a perfect matching, then  $D = A = \emptyset$ , and  $G[C] = G$ . The Edmonds-Gallai decomposition of a graph can be obtained efficiently by using Edmonds' matching algorithm [4].

Before stating the Edmonds-Gallai Structure Theorem, we need the following definitions. A graph  $H$  is said to be **factor-critical** if deleting any vertex from  $H$  results in a graph with a perfect matching. Such a matching in  $H$  is called **near-perfect**. The primary importance of the Edmonds-Gallai decomposition is contained in the following theorem.

**Theorem 1.3** (*Edmonds-Gallai Structure Theorem*) *Let  $G$  be a graph and  $D \cup A \cup C$  be the Edmonds-Gallai decomposition of  $G$ . Then  $A$  is a Tutte set,  $G[D]$  is the union of the odd components of  $G - A$ , each of which is factor-critical, and  $G[C]$  is the union of the even components of  $G - A$ . Moreover, any maximum matching in  $G$  consists of*

- *a perfect matching in  $G[C]$ ;*
- *a near-perfect matching in every (odd) component of  $G[D]$ ;*
- *an edge joining  $v$  to some vertex in  $D$ , for every  $v \in A$ .*

The Edmonds-Gallai decomposition of  $G$  is closely related to the structure of maximal Tutte sets in  $G$ . Indeed [7], the set  $A$  is the intersection of all the maximal Tutte sets in  $G$ , and no vertex in the set  $D$  can occur in any Tutte set of  $G$ . In fact, we have (cf. Theorem 3.5 in [1])

**Theorem 1.4** *Let  $G$  be a graph, and  $X \subseteq V(G)$ . Then  $X$  is a maximal Tutte set in  $G$  if and only if  $X = A \cup Z$ , where  $Z$  is a maximal Tutte set in  $G[C]$ .*

Since  $G[C]$  always contains a perfect matching [7], this shows that finding maximal Tutte sets in  $G$  reduces to finding maximal Tutte sets in graphs which contain a perfect matching. In the sequel, therefore, we will focus on the complexity of finding a maximal Tutte set in a graph with a perfect matching.

In [1], we found that the study of maximal Tutte sets in a graph  $G$  with a perfect matching is greatly facilitated by introducing a related graph  $D(G)$ . When  $G$  contains a perfect matching, we define  $D(G)$  as follows:  $V(D(G)) = V(G)$ , and  $E(D(G)) = \{(x, y) \mid G - \{x, y\} \text{ contains a perfect matching}\}$ . We call a graph  $H$  a  $D$ -graph if  $H = D(G)$  for some graph  $G$ .

There is a useful alternative definition of  $E(D(G))$ . Let  $M$  be a perfect matching in  $G$ . We denote by  $P_M[x, y]$  an  $M$ -alternating-path in  $G$  joining  $x$  and  $y$ , which begins and ends with an edge in  $M$ . Similarly, we denote by  $P_M(x, y)$  an  $M$ -alternating-path in  $G$  joining  $x$  and  $y$ , which begins and ends with an edge not in  $M$ ; the  $M$ -alternating-paths  $P_M[x, y]$  and  $P_M(x, y)$  are defined analogously. By a theorem of Berge [2],  $(x, y) \in E(D(G))$  if and only if there exists a path  $P_M[x, y]$  in  $G$ . Clearly, this definition of  $E(D(G))$  is independent of the choice of the perfect matching  $M$ .

A key result for this paper is the following (cf. Theorem 3.4 in [1]).

**Theorem 1.5** *Let  $G$  be a graph with a perfect matching, and let  $X \subseteq V(G)$ . Then  $X$  is a maximal Tutte set in  $G$  if and only if  $X$  is a maximal independent set in  $D(G)$ .*

Let  $M$  be a perfect matching in  $G$  and  $x \in V(G)$ . We denote by  $x'$  the vertex in  $V(G)$  that is matched to  $x$  under  $M$ . We note that if  $(x, y) \in E(G) - M$ , then  $(x', y') \in E(D(G))$  since  $G$  contains the 3-path  $P_M[x', y'] = (x', x, y, y')$ . Thus  $G$  is isomorphic to a spanning subgraph of  $D(G)$  via the mapping from  $V(G)$  to  $V(D(G))$  given by  $x \rightarrow x'$ . We denote this fact by  $G \preceq D(G)$ .

We define the iterated  $D$ -graphs of  $G$  recursively as follows:  $D^0(G) = G$ , and  $D^k(G) = D(D^{k-1}(G))$ , for  $k \geq 1$ . Since  $V(D(G)) = V(G)$  and  $G \preceq D(G)$ , it follows that for any graph  $G$ , there exists an integer  $l \geq 0$  such that  $D^l(G) \cong D^{l+1}(G)$ . We call the smallest such integer  $l$  the **level** of  $G$  and denote it by  $\text{level}(G)$ . In [1], we prove the following unexpected result.

**Theorem 1.6** *For any graph  $G$ ,  $\text{level}(G) \leq 2$ .*

The graphs  $G$  of level 0 (i.e., with  $G \cong D(G)$ ) will be of special interest. In [1], we characterize such graphs using the following definition. Let  $G$  be a graph with a perfect matching  $M$ . We say  $G$  has the  **$C_4$ -property** if and only if whenever  $G$  contains a path  $P_M(x, y)$  of length 3,  $G$  also contains the edge  $(x, y)$ .

In [1] we prove the following; for convenience, we include the proof here.

**Theorem 1.7** *Let  $G$  be a graph with a perfect matching. Then  $G$  has level 0 if and only if  $G$  satisfies the  $C_4$ -property.*

PROOF: Let  $M$  be a perfect matching in  $G$ .

( $\Rightarrow$ ) Suppose  $G$  contains a path  $P_M(x, y)$  of length 3. If  $[x, y] \in M$  we are done. Else  $[x', x] \circ P_M(x, y) \circ [y, y']$  is a path  $P_M[x', y']$  in  $G$  of length 5, and  $(x', y') \in E(D(G))$ . Since  $G \cong D(G)$  via the mapping  $x \rightarrow x'$ , it follows that  $(x, y) \in E(G)$ . Thus  $G$  has the  $C_4$ -property.

( $\Leftarrow$ ) Let  $(u', v') \in E(D(G))$ . It suffices to show that  $(u, v) \in E(G)$ . Since  $(u', v') \in E(D(G))$ , there exists a path  $P_M[u', v']$  in  $G$ , and thus a path  $P_M(u, v) = (u, a_1, a'_1, a_2, a'_2, \dots, a_r, a'_r, v)$  in  $G$ . Since  $G$  has the  $C_4$ -property, we obtain by a simple iterative argument that  $G$  contains the edges

$(u, a_2), (u, a_3), \dots, (u, a_r), (u, v)$ .  $\square$

## 2 Main Results

We now summarize the remainder of this paper. In Section 2.1, we give two efficient algorithms to construct maximal Tutte sets in graphs. In Section 2.2, we show that it is NP-complete to find maximum Tutte sets in general graphs, and that it remains NP-complete for the class of planar graphs,  $k$ -connected graphs, and triangle-free graphs. By contrast, we show in Section 2.3 that maximum Tutte sets can be found in polynomial time for graphs having level 0 or 1, elementary graphs, and 1-tough graphs. We conclude in Section 3 with a short discussion of some open questions.

### 2.1 Efficient Algorithms for Maximal Tutte Sets

We now present two efficient algorithms to find a maximal Tutte set in a graph.

The first algorithm uses Theorems 1.4 and 1.5. Given  $G$ , find the Edmonds-Gallai decomposition  $D \cup A \cup C$  of  $V(G)$ . Construct  $D(G[C])$ , and find a maximal independent set  $Z$  in  $D(G[C])$ . By Theorems 1.4 and 1.5,  $Z$  is a maximal Tutte set in  $G[C]$  and  $A \cup Z$  is a maximal Tutte set in  $G$ .

The second algorithm is based directly on the Edmonds-Gallai Structure Theorem.

**Algorithm: Maximal Tutte set in  $G$**

Let  $D_0 \cup A_0 \cup C_0$  be the Edmonds-Gallai decomposition of  $V(G)$ ;

$X := A_0$ ;  $G_0 := G[C_0]$ ;  $i := 0$ ;

**while**  $C_i \neq \emptyset$  **do**

  let  $v_i$  be an arbitrary vertex in  $C_i$ ;

let  $D_{i+1} \cup A_{i+1} \cup C_{i+1}$  be the Edmonds-Gallai decomposition of  $V(G_i - v_i)$ ;

$X := X \cup A_{i+1} \cup \{v_i\}$ ;  $G_{i+1} := G[C_i]$ ;  $i := i + 1$ ;

**return**  $X$

To see that the algorithm is correct, note that  $E(C_i, D_i) = \emptyset$  (by definition) and  $D_{i+1} \subseteq C_i$ . Thus  $E(D_i, D_j) = \emptyset$  for  $i \neq j$ . Since the components of  $G[D_i]$ ,  $i \geq 0$ , are factor-critical by the Edmonds-Gallai Structure Theorem, it follows that all the components of  $G - X = G[D_0 \cup D_1 \cup \dots]$  are also factor-critical, and so  $X$  is a maximal Tutte set in  $G$ .

It can be shown that the set  $X$  returned by the second algorithm will be a maximum Tutte set of  $G$  if and only if  $v_i \in C_i$  is always selected to be a vertex which occurs in a maximum Tutte set of  $G[C_i]$ . But as we are about to see, selecting such a vertex is almost certainly intractible.

## 2.2 Finding a Maximum Tutte Set is NP-complete

Consider the following decision problem.

### MAX TUTTE SET

INSTANCE: Graph  $G$ , integer  $k \geq 0$ .

QUESTION: Does  $G$  contain a Tutte set  $X$  with  $|X| \geq k$ ?

**Theorem 2.1** *MAX TUTTE SET is NP-complete.*

PROOF: Clearly MAX TUTTE SET  $\in NP$ , and we only show it is NP-hard.

We will use a polynomial reduction from the following well-known NP-complete problem [5].

### INDEPENDENT SET

INSTANCE: Graph  $G$ , integer  $k \geq 0$ .

QUESTION: Is  $\alpha(G) \geq k$ ?

Let  $(H, k)$  be any instance of INDEPENDENT SET, where  $V(H) = \{v_1, v_2, \dots, v_n\}$ . Form the graph  $H'$  by attaching a graph  $S_i$  to each vertex  $v_i$ , where  $S_i$  consists of a  $C_4 : v_i, a_{i,1}, a_{i,2}, a_{i,3}, v_i$  with one chord  $(a_{i,1}, a_{i,3})$ . One easily checks that  $H'$  contains a perfect matching. Concerning  $D(H')$ , note that

1.  $D(S_i)$  is complete, except for the edge  $(a_{i,1}, a_{i,3})$ ;
2.  $V(H)$  is an independent set in  $D(H')$ ;

3. If  $(v_i, v_j) \in E(H)$ , there is a complete bipartite join between  $D(S_i) - v_i$  and  $D(S_j) - v_j$  in  $D(H')$ ;
4. If  $(v_i, v_j) \notin E(H)$ , there are no edges in  $D(H')$  joining  $D(S_i)$  and  $D(S_j)$ .

Let  $m(H')$  denote the cardinality of a maximum Tutte set in  $H'$ . Using the above observations and Theorem 1.5, it is now easy to see that

$$m(H') = \alpha(D(H')) = |V(H)| + \alpha(H).$$

This completes the polynomial reduction.  $\square$

It is interesting to consider the complexity of MAX TUTTE SET for  $k$ -connected graphs, where  $k \geq 2$ . We note that trivially INDEPENDENT SET remains NP-complete for connected graphs.

**Theorem 2.2** *MAX TUTTE SET is NP-complete for  $k$ -connected graphs, for any  $k \geq 2$ .*

PROOF: The proof follows the same lines as the proof of Theorem 2.1. Given  $H$  we now construct  $H'$  as follows. For every  $v_i \in V(H)$ , we attach the graph  $S_i$  as shown in Figure 1. In Figure 1, each circle represents a set of  $k$  independent vertices, where  $V_i \in S_i$  is identified with  $v_i \in H$ . The double edge connecting sets of vertices represents a complete bipartite join. To complete the construction of  $H'$ , we connect  $V_i$  to  $V_j$  by a complete bipartite join if and only if  $(v_i, v_j) \in E(H)$ .

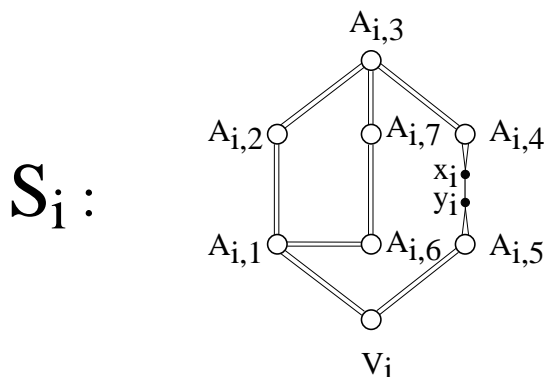


Figure 1: Each circle represents  $k$  independent vertices, and each double edge represents a complete bipartite join.

It is easy to see that if  $H$  is connected, then  $H'$  is  $k$ -connected. Since each  $S_i$  has a perfect matching, so does  $H'$ . It can be shown that no perfect matching in  $H'$  can contain an edge joining a vertex of  $V_i$  to a vertex of  $V_j$ . Note that  $D(H')$  has properties analogous to (1)-(4) in Theorem 2.1. Since  $\{x_i\} \cup \bigcup_{k=0}^3 A_{i,2k+1}$  forms a

maximum independent set in  $D(S_i)$ , and  $\{y_i\} \cup V_i$  are also independent in  $D(S_i)$ , we have

$$\begin{aligned} m(H') = \alpha(D(H')) &= \alpha(H)(4k + 1) + (|V(H)| - \alpha(H))(k + 1) \\ &= (k + 1)|V(H)| + 3k\alpha(H), \end{aligned}$$

completing the polynomial reduction.  $\square$

Now suppose that  $H$  is a 2-connected planar graph with a perfect matching  $M$ , and let  $S_i$  be as in Figure 1 with  $k = 1$ . Construct  $H'$  as in the proof of Theorem 2.2, but add an edge between vertices  $A_{i,4}$  and  $A_{j,4}$  precisely if  $(v_i, v_j) \in M$ . The resulting graph  $H'$  will be planar and 2-connected. Moreover,  $D(H')$  has the same properties as  $D(H)$  in the proof of Theorem 2.2, with the exception that now  $\{y_i\} \cup V_i \cup A_{i,4}$  are independent in  $D(S_i)$ . Hence we have

$$\begin{aligned} m(H') = \alpha(D(H')) &= 5\alpha(H) + 3(|V(H)| - \alpha(H)) \\ &= 3|V(H)| + 2\alpha(H). \end{aligned}$$

Since INDEPENDENT SET is NP-complete for the class of 2-connected planar graphs with a perfect matching [6], we have the following result.

**Theorem 2.3** *MAX TUTTE SET is NP-complete for the class of 2-connected planar graphs.*

In the proof of Theorem 2.2, note that if  $H$  is triangle-free, then so is  $H'$ . Since INDEPENDENT SET remains NP-complete for triangle-free graphs [5], we have:

**Theorem 2.4** *MAX TUTTE SET is NP-complete for triangle-free graphs.*

## 2.3 Classes of Graphs for which MAX TUTTE SET can be Solved in Polynomial Time

In contrast to the NP-completeness results of Section 2.2, we now consider several interesting classes of graphs in which maximum Tutte sets can be found in polynomial time.

### 2.3.1 Graphs with Level 0 or 1

We will prove

**Theorem 2.5** *MAX TUTTE SET  $\in P$  for the class of graphs with level 0 or 1.*

In order to prove this, we first require

**Lemma 2.6** *INDEPENDENT SET  $\in P$  for the class of level 0 graphs.*



PROOF OF LEMMA 2.6: Let  $G$  be a graph with  $\text{level}(G) = 0$ , and let  $M$  be a perfect matching in  $G$ . Let  $I_0$  be an independent set in  $G$ , and let  $S_0 = I_0 \cup I'_0$ , where  $I'_0$  consists of the mates of the vertices of  $I_0$  under  $M$ . Note that  $S_0$  can be partitioned into  $|I_0|$  sets, each of which induces a clique whose vertices are perfectly matched under  $M$ .

Suppose now that  $I \subseteq S \subseteq V(G)$ , where

1.  $I$  is independent in  $G$ ;
2.  $S$  can be partitioned into  $|I|$  sets, each of which induces a clique whose vertices are perfectly matched under  $M$ .

If  $S = V(G)$ , then clearly  $I$  is a maximum independent set in  $G$ . Otherwise, there exists an edge  $(v, v') \in M$ , with  $v, v' \notin S$ . If  $v$  and  $v'$  are adjacent, respectively, to distinct vertices  $x, y \in I$ , then by the  $C_4$ -property,  $x$  must be adjacent to  $y$ , contradicting the independence of  $I$ . Thus there cannot be two independent edges between two vertices in  $I$  and  $\{v, v'\}$ .

We now consider two cases, indicating in each how to redefine  $I$  and  $S$  so that  $|S|$  increases, and (1) and (2) still hold. By iterating this procedure, we eventually obtain  $S = V(G)$ , at which point  $I$  will be a maximum independent set in  $G$ .

*Case 1:*  $v$  (resp.,  $v'$ ) is not adjacent to any vertex in  $I$ .

Redefine  $I$  to be  $I \cup \{v\}$  (resp.,  $I \cup \{v'\}$ ), and  $S$  to be  $S \cup \{v, v'\}$ . It is easy to see that (1) and (2) still hold.

*Case 2:* There exists a unique vertex  $x \in I$  such that  $(x, v), (x, v') \in E(G)$ .

Let  $K$  be the even clique in the partition of  $V(S)$  that contains  $x$ , and let  $y$  be any vertex in  $K - x$ . Of course,  $x$  and  $y$  are matched in some perfect matching in  $K$ . If  $y$  were adjacent to a vertex  $z \in I - x$ , then by the  $C_4$ -property,  $v$  and  $v'$  are each adjacent to  $z$ , contradicting the uniqueness of  $x \in I$  as a neighbor of both  $v$  and  $v'$ . Thus  $I_y = (I - x) \cup \{y\}$  is an independent set in  $G$  for any  $y \in K - x$ , with  $|I_y| = |I|$ .

Since  $(x, v), (x, v') \in E(G)$ ,  $y$  is the only vertex in  $I_y$  that might be adjacent to either  $v$  or  $v'$ . If  $v$  (resp.,  $v'$ ) is not adjacent to  $y$ , then redefine  $I$  to be  $I_y \cup \{v\}$  (resp.,  $I_y \cup \{v'\}$ ), and  $S$  to be  $S \cup \{v, v'\}$ , observing that (1) and (2) still hold. But if  $v, v'$  are each adjacent to each  $y \in K$ , then  $K \cup \{v, v'\}$  is an even clique. Leaving  $I$  unchanged, but redefining  $S$  to be  $S \cup \{v, v'\}$ , we see that (1) and (2) still hold.

This proves Lemma 2.6.  $\square$

PROOF OF THEOREM 2.5: Let  $M$  be a perfect matching in  $G$ .

If  $\text{level}(G) = 0$ , let  $I$  be a maximum independent set in  $G$ . By Lemma 2.6, we can obtain  $I$  in polynomial time. Since  $G \cong D(G)$  and  $G \preceq D(G)$ , we have that  $I' = \{x' \mid x \in I\}$  is a maximum independent set in  $D(G)$ . But then by Theorem 1.5,  $I'$  is a maximum Tutte set in  $G$ .

If  $\text{level}(G) = 1$ , then  $\text{level}(D(G)) = 0$ . By Lemma 2.6, we can construct a maximum independent set  $I$  in  $D(G)$  in polynomial time. By Theorem 1.5,  $I$  is a maximum Tutte set in  $G$ .  $\square$

### 2.3.2 Elementary Graphs

A graph  $G$  is called **elementary** if it contains a perfect matching and if the edges which occur in a perfect matching in  $G$  induce a connected subgraph. A substantial study of elementary graphs and their properties is given in [7], where the following result is proved.

**Theorem 2.7** *Let  $G$  be a graph with a perfect matching. Then  $G$  is elementary if and only if  $G$  satisfies any of the following conditions:*

- (i) *the maximal Tutte sets in  $G$  form a partition of  $V(G)$ ;*
- (ii)  *$C_{G-x} = \emptyset$ , for all  $x \in V(G)$ ;*
- (iii) *for any non-empty Tutte set  $X \subseteq V(G)$ ,  $G - X$  has only odd components.*

The following is also proved in [7] (cf. Theorem 5.2.2(b)).

**Theorem 2.8** *Let  $G$  be an elementary graph, with  $x, y \in V(G)$ . Then  $G - \{x, y\}$  has a perfect matching if and only if  $x$  and  $y$  occur in different maximal Tutte sets in  $G$ .*

Theorems 2.7(i) and 2.8, together with the definition of  $D(G)$ , immediately give the following.

**Theorem 2.9** *A graph  $G$  is elementary if and only if  $D(G)$  is a complete multipartite graph.*

Since finding a maximum independent set in a complete multipartite graph is trivial, Theorem 1.5 immediately yields the following result.

**Theorem 2.10**  *$\text{MAX TUTTE SET} \in P$  for the class of elementary graphs.*

### 2.3.3 1-Tough Graphs

A graph  $G$  is called **1-tough** if  $\omega(G - X) \leq |X|$  for all non-empty  $X \subseteq V(G)$ . We wish to consider the complexity of finding maximum Tutte sets in 1-tough graphs. To this end, we now prove two theorems.

**Theorem 2.11** *If  $G$  is 1-tough on an odd number of vertices, then  $G$  is factor-critical.*

PROOF: Suppose  $G$  is 1-tough on an odd number of vertices, but not factor-critical. Then there exists  $v \in V(G)$  such that  $G' = G - v$  has no perfect matching. Thus there exists  $X' \subseteq V(G')$  with  $\omega_0(G' - X') = |X'| + 1 + k$ , where  $k \geq 0$ . Setting  $X = X' \cup \{v\}$ , we have

$$\begin{aligned}
\omega(G - X) &= \omega_0(G - X) + \omega_e(G - X) \\
&= \omega_0(G' - X') + \omega_e(G' - X') \\
&= |X'| + 1 + k + \omega_e(G' - X') \\
&= |X| + k + \omega_e(G' - X') \\
&\geq |X| \geq 1.
\end{aligned}$$

Since  $G$  is 1-tough,  $k = \omega_e(G' - X') = 0$ ; otherwise  $\omega(G - X) > |X| \geq 1$ , a contradiction. Letting  $H_1, \dots, H_{|X'|+1}$  denote the odd components of  $G' - X'$ , we find  $|V(G)| = 1 + |X'| + \sum_{i=1}^{|X'|+1} |V(H_i)|$  is even, a contradiction.  $\square$

**Theorem 2.12** *If  $G$  is 1-tough on an even number of vertices, then  $G$  is elementary.*

PROOF: Clearly  $G$  contains a perfect matching. If  $G$  is not elementary, then by Theorem 2.7(iii),  $G$  would contain a nonempty Tutte set  $X$  such that  $G - X$  contains one or more even components. Since  $X$  is a Tutte set,  $G - X$  contains at least  $|X|$  odd components as well. Thus  $\omega(G - X) \geq |X| + 1 > |X| \geq 1$ , and  $G$  would not be 1-tough a contradiction.  $\square$

From Theorems 2.11 and 2.12 we have the following.

**Theorem 2.13** *MAX TUTTE SET  $\in P$  for the class of 1-tough graphs.*

PROOF: Let  $G$  be a 1-tough graph on  $n$  vertices. If  $n$  is odd, then  $G$  is factor-critical by Theorem 2.11, and thus the only Tutte set in  $G$  is the empty set. If  $n$  is even, then  $G$  is elementary by Theorem 2.12, and we can find a maximum Tutte set in  $G$  in polynomial time by Theorem 2.10.  $\square$

**Corollary 2.14** *MAX TUTTE SET  $\in P$  for the following classes of graphs:*

- (a) *hamiltonian graphs,*
- (b) *2-connected claw-free graphs,*
- (c)  *$k$ -regular,  $k$ -edge-connected graphs, for any  $k \geq 1$ .*

PROOF: It is well-known that hamiltonian graphs and 2-connected claw-free graphs are 1-tough. It is also easy to show that  $k$ -regular,  $k$ -edge-connected graphs are 1-tough for any  $k \geq 1$ .  $\square$

### 3 Open problems

We conclude with several open problems.

1. We showed that MAX TUTTE SET can be solved in polynomial time for graphs of level 0 or 1, elementary graphs, and 1-tough graphs. Are there other interesting classes of graphs for which MAX TUTTE SET can be solved in polynomial time?
2. We know that MAX TUTTE SET is NP-complete for 2-connected planar graphs (Theorem 2.3) and polynomial for 4-connected planar graphs, since they are hamiltonian (Corollary 2.14). What is the complexity of MAX TUTTE SET for 3-connected planar graphs?
3. By Theorem 2.13, MAX TUTTE SET can be solved in polynomial time for 1-tough graphs, and hence for planar 1-tough graphs. Given  $\epsilon > 0$ , is MAX TUTTE SET polynomial for planar  $(1 - \epsilon)$ -tough graphs?

We strongly believe that MAX TUTTE SET is NP-complete for  $(1 - \epsilon)$ -tough general graphs. A possible approach to proving this is to first note that INDEPENDENT SET remains NP-complete for the class of hamiltonian graphs. If  $H$  in the proof of Theorem 2.2 is hamiltonian, the resulting graph  $H'$  appears to have toughness  $\frac{k+1}{k+2}$ , with tough set  $V_1 \cup \{y_1\}$ .

4. The class of  $D$ -graphs has been useful in our study of Tutte sets. But it remains an open problem whether level 1  $D$ -graphs can be recognized in polynomial time (the problem is uninteresting for level 0 or 2 graphs, of course). This recognition problem becomes trivial for the class of bipartite graphs, since it was proved in [1] that a bipartite graph  $G$  is a  $D$ -graph if and only if  $\text{level}(G) = 0$ , and thus there are no bipartite level 1  $D$ -graphs.

### Acknowledgment

The authors gratefully acknowledge several helpful conversations with Luca Bussolari and Naomi Russo during the preparation of this manuscript.

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