Abstract

We study theorems giving sufficient conditions on the vertex degrees of a graph $G$ to guarantee $G$ is $t$-tough. We first give a best monotone theorem when $t \geq 1$, but then show that for any integer $k \geq 1$, a best monotone theorem for $t = \frac{1}{k} \leq 1$ requires at least $f(k) \cdot |V(G)|$ nonredundant conditions, where $f(k)$ grows superpolynomially as $k \to \infty$. When $t < 1$, we give an additional, simple theorem for $G$ to be $t$-tough, in terms of its vertex degrees.

1 Introduction

We consider only simple graphs without loops or multiple edges. Our terminology and notation will be standard except as indicated, and a good reference for any undefined terms or notation is [7]. For two graphs $G, H$ on disjoint vertex sets, we
denote their union by $G \cup H$. The join $G + H$ of $G$ and $H$ is the graph formed from $G \cup H$ by adding all edges between $V(G)$ and $V(H)$.

For a positive integer $n$, an $n$-sequence (or just a sequence) is an integer sequence $\pi = (d_1, d_2, \ldots, d_n)$, with $0 \leq d_j \leq n - 1$ for all $j$. In contrast to [7], we will usually write the sequence in nondecreasing order (and may make this explicit by writing $\pi = (d_1 \leq \cdots \leq d_n)$). We will employ the standard abbreviated notation for sequences, e.g., $(4, 4, 4, 4, 5, 5, 6)$ will be denoted $4^5 5^2 6^1$. If $\pi = (d_1, \ldots, d_n)$ and $\pi' = (d_1', \ldots, d_n')$ are two $n$-sequences, we say $\pi'$ majorizes $\pi$, denoted $\pi' \geq \pi$, if $d_j' \geq d_j$ for all $j$.

A degree sequence of a graph is any sequence $\pi = (d_1, d_2, \ldots, d_n)$ consisting of the vertex degrees of the graph. A sequence $\pi$ is graphical if there exists a graph $G$ having $\pi$ as one of its degree sequences, in which case we call $G$ a realization of $\pi$. If $P$ is a graph property (e.g., hamiltonian, $k$-connected), we call a graphical sequence $\pi$ forcibly $P$ if every realization of $\pi$ has property $P$.

Historically, the degree sequence of a graph has been used to provide sufficient conditions for a graph to have certain properties, such as hamiltonicity or $k$-connectivity. In particular, sufficient conditions for $\pi$ to be forcibly hamiltonian were given by several authors, culminating in the following theorem of Chvátal [4].

**Theorem 1.1** ([4]). Let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence, with $n \geq 3$. If $d_i \leq i < \frac{1}{2}n$ implies $d_{n-i} \geq n-i$, then $\pi$ is forcibly hamiltonian.

Unlike its predecessors, Chvátal’s theorem has the property that if it does not guarantee that $\pi$ is forcibly hamiltonian because the condition fails for some $i < \frac{1}{2}n$, then $\pi$ is majorized by $\pi' = i^i (n-i-1)^{n-2i} (n-1)^i$, which has a unique non-hamiltonian realization $K_i + (K_i \cup K_{n-2i})$. As we will see below, this implies that Chvátal’s theorem is the strongest of an entire class of theorems giving sufficient degree conditions for $\pi$ to be forcibly hamiltonian.

Sufficient conditions for $\pi$ to be forcibly $k$-connected were given by several authors, culminating in the following theorem of Bondy [3] (though the form in which we present it is due to Boesch [2]).

**Theorem 1.2** ([2, 3]). Let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence with $n \geq 2$, and let $1 \leq k \leq n-1$. If $d_i \leq i + k - 2$ implies $d_{n-k+1} \geq n-i$, for $1 \leq i \leq \frac{1}{2} (n-k+1)$, then $\pi$ is forcibly $k$-connected.

Boesch [2] also observed that Theorem 1.2 is the strongest theorem giving sufficient degree conditions for $\pi$ to be forcibly $k$-connected, in exactly the same sense as Theorem 1.1.

Let $\omega(G)$ denote the number of components of a graph $G$. For $t \geq 0$, we call $G$ $t$-tough if $t \cdot \omega(G - X) \leq |X|$, for every $X \subseteq V(G)$ with $\omega(G - X) > 1$. The
toughness of $G$, denoted $\tau(G)$, is the maximum $t \geq 0$ for which $G$ is $t$-tough (taking $\tau(K_n) = n - 1$, for all $n \geq 1$). So if $G$ is not complete, then

$$\tau(G) = \min \left\{ \frac{|X|}{\omega(G - X)} \mid X \subseteq V(G) \text{ is a cutset of } G \right\}.$$ 

In this paper we consider forcibly $t$-tough theorems, for any $t \geq 0$. When trying to formulate and prove this type of theorem, we encountered very different behavior in the number of conditions required for a best possible theorem for the cases $t \geq 1$ and $t < 1$. In order to describe this behavior precisely, we need to say what we mean by a ‘condition’ and by a ‘best possible theorem’.

First note that the conditions in Theorems 1.1 can be written in the form:

$$d_i \geq i + 1 \text{ or } d_{n-i} \geq n - i, \text{ for } i = 1, \ldots, \left\lfloor \frac{1}{2} (n - 1) \right\rfloor,$$

and the conditions in Theorem 1.2 can be written in a similar way. We will use the term ‘Chvátal-type conditions’ for such conditions. Formally, a Chvátal-type condition for $n$-sequences $(d_1 \leq d_2 \leq \cdots \leq d_n)$ is a condition of the form

$$d_{i_1} \geq k_{i_1} \lor d_{i_2} \geq k_{i_2} \lor \cdots \lor d_{i_r} \geq k_{i_r},$$

where all $i_j$ and $k_{i_j}$ are integers, with $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ and $1 \leq k_{i_1} \leq k_{i_2} \leq \cdots \leq k_{i_r} \leq n$.

A graph property $P$ is called increasing if whenever a graph $G$ has $P$, so has every edge-augmented supergraph of $G$. In particular, “hamiltonian”, “$k$-connected” and “$t$-tough” are all increasing graph properties. In this paper, the term “graph property” will always mean an increasing graph property.

Given a graph property $P$, consider a theorem $T$ which declares certain degree sequences to be forcibly $P$, rendering no decision on the remaining degree sequences. We call such a theorem $T$ a forcibly $P$-theorem (or just a $P$-theorem, for brevity). Thus Theorem 1.1 would be a forcibly hamiltonian theorem. We call a $P$-theorem $T$ monotone if, for any two degree sequences $\pi, \pi'$, whenever $T$ declares $\pi$ forcibly $P$ and $\pi' \geq \pi$, then $T$ declares $\pi'$ forcibly $P$. We call a $P$-theorem $T$ optimal if whenever $T$ does not declare a degree sequence $\pi$ forcibly $P$, then $\pi$ is not forcibly $P$; $T$ is weakly optimal if for any sequence $\pi$ (not necessarily graphical) which $T$ does not declare forcibly $P$, $\pi$ is majorized by a degree sequence which is not forcibly $P$.

A $P$-theorem which is both monotone and weakly optimal is a best monotone $P$-theorem, in the following sense.

**Theorem 1.3.** Let $T, T_0$ be monotone $P$-theorems, with $T_0$ weakly optimal. If $T$ declares a degree sequence $\pi$ to be forcibly $P$, then so does $T_0$.

**Proof of Theorem 1.3:** Suppose to the contrary that there exists a degree sequence $\pi$ so that $T$ declares $\pi$ forcibly $P$, but $T_0$ does not. Since $T_0$ is weakly
optimal, there exists a degree sequence $\pi' \geq \pi$ which is not forcibly $P$. This means that also $T$ will not declare $\pi'$ forcibly $P$. But if $T$ declares $\pi$ forcibly $P$, $\pi' \geq \pi$, and $T$ does not declare $\pi'$ forcibly $P$, then $T$ is not monotone, a contradiction. 

If $T_0$ is Chvátal’s hamiltonian theorem (Theorem 1.1), then $T_0$ is clearly monotone, and we noted above that $T_0$ is weakly optimal. So by Theorem 1.3, Chvátal’s theorem is a best monotone hamiltonian theorem.

Our goal in this paper is to consider forcibly $t$-tough theorems, for any $t \geq 0$. In Section 2 we first give a best monotone $t$-tough theorem for $n$-sequences, requiring at most $\left\lceil \frac{1}{2} n \right\rceil$ Chvátal-type conditions, for any $t \geq 1$. In contrast to this, in Sections 3 and 4 we show that for any integer $k \geq 1$, a best monotone $1/k$-tough theorem contains at least $f(k) \cdot n$ nonredundant Chvátal-type conditions, where $f(k)$ grows superpolynomially as $k \to \infty$. A similar superpolynomial growth in the complexity of the best monotone $k$-edge-connected theorem in terms of $k$ was previously noted by Kriesell [6].

This superpolynomial complexity of a best monotone $1/k$-tough theorem suggests the desirability of finding more reasonable $t$-tough theorems, when $t < 1$. In Section 5 we give one such theorem. This theorem is a monotone, though not best monotone, $t$-tough theorem which is valid for any $t \leq 1$.

## 2 A Best Monotone $t$-Tough Theorem for $t \geq 1$

We first give a best monotone $t$-tough theorem for $t \geq 1$.

**Theorem 2.1.** Let $t \geq 1$, $n \geq \lceil t \rceil + 2$, and let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence. If

$$
(*t) \quad d_{\lceil i/t \rceil} \geq i + 1 \quad \text{or} \quad d_{n-i} \geq n - \lceil i/t \rceil, \quad \text{for} \quad t \leq i < \frac{tn}{(t+1)},
$$

then $\pi$ is forcibly $t$-tough.

Clearly, property $(*t)$ in Theorem 2.1 is monotone. Furthermore, if $\pi$ does not satisfy $(*t)$ for some $i$ with $t \leq i < tn/(t+1)$, then $\pi$ is majorized by $\pi' = (\beta^{\lceil i/t \rceil}_{\beta}) \cdot (n - \lceil i/t \rceil - 1)^{\alpha - \lceil i/t \rceil} \cdot (n-1)^{\beta}$, which has the non-$t$-tough realization $K_i + \tilde{K}_{\lceil i/t \rceil} \cup K_{n-i - \lceil i/t \rceil}$. Thus $(*t)$ in Theorem 2.1 is also weakly optimal, and so Theorem 2.1 is best monotone by Theorem 1.3. Finally, note that when $t = 1$, $(*t)$ reduces to Chvátal’s hamiltonian condition in Theorem 1.1.

**Proof of Theorem 2.1:** Suppose $\pi$ satisfies $(*t)$ for some $t \geq 1$ and $n \geq \lceil t \rceil + 2$, but $\pi$ has a realization $G$ which is not $t$-tough. Then there exists a set $X \subseteq V(G)$ that is maximal with respect to $\omega(G-X) \geq 2$ and $\frac{|X|}{\omega(G-X)} < t$. Let $x = |X|$
and \( w = \omega(G - X) \), so that \( w \geq \lfloor x/t \rfloor + 1 \). Also, let \( H_1, H_2, \ldots, H_w \) denote the components of \( G - X \), with \( |H_1| \geq |H_2| \geq \cdots \geq |H_w| \), and let \( h_j = |H_j| \) for \( j = 1, \ldots, w \). By adding edges (if needed) to \( G \), we may assume \( \langle X \rangle \) is complete, and each \( \langle H_j \rangle \) is complete and completely joined to \( X \).

Set \( i = x + h_2 - 1 \).

**Claim 1.** \( i \geq t \).

**Proof:** It is enough to show that \( x \geq t \). Assume instead that \( x < t \). Define \( X' = X \cup \{v\} \), with \( v \in H_1 \). If \( h_1 \geq 2 \), then

\[
\frac{|X'|}{\omega(G - X')} = \frac{x + 1}{\omega(G - X)} < \frac{t + 1}{2} \leq t,
\]

which contradicts the maximality of \( X \). Similarly, if \( h_1 = 1 \) and \( w \geq 3 \), then

\[
\frac{|X'|}{\omega(G - X')} = \frac{x + 1}{\omega(G - X) - 1} < \frac{t + 1}{2} \leq t,
\]

also a contradiction. Finally, if \( h_1 = 1 \) and \( w = 2 \), then \( G \) is the graph \( K_{n-2} + \overline{K_2} \) with \( n - 2 = x < t \), contradicting \( n \geq \lceil t \rceil + 2 \).

**Claim 2.** \( i < \frac{tn}{t+1} \)

**Proof:** Note that \( n = x + h_1 + h_2 + \cdots + h_w \geq x + 2h_2 + w - 2 \). Since \( x < tw \), we obtain

\[
(t + 1)i = (t + 1)(x + h_2 - 1) = t(x + h_2 - 1) + x + h_2 - 1 < t(x + h_2 - 1) + tw + t(h_2 - 1) \leq tn.
\]

By the claims we have \( t \leq i < \frac{tn}{t+1} \). Next note that

\[
\left\lfloor \frac{i}{t} \right\rfloor = \left\lfloor \frac{x + h_2 - 1}{t} \right\rfloor \leq \left\lfloor \frac{x}{t} \right\rfloor + h_2 - 1 \leq w + h_2 - 2 \leq \sum_{j=2}^{w} h_j = n - x - h_1,
\]

so

\[
d_{\lfloor i/t \rfloor} \leq d_{n-x-h_1} = x + h_2 - 1 = i.
\]

However, we also have

\[
d_{n-i} \leq d_{n-x} = x + h_1 - 1 = n - h_2 - (h_3 + \cdots + h_w - 1) \leq n - (w + h_2 - 1) < n - \left( \frac{x}{t} + h_2 - 1 \right) \leq n - \frac{x + h_2 - 1}{t} = n - i/t \leq n - \left\lfloor i/t \right\rfloor,
\]

contradicting \((st)\).

\[\square\]
The Number of Chvátal-Type Conditions in Best Monotone Theorems

In this section we provide a theory that allows us to lower bound the number of degree sequence conditions required in a best monotone $P$-theorem. 

Recall that a Chvátal-type condition for $n$-sequences $(d_1 \leq d_2 \leq \cdots \leq d_n)$ is a condition of the form

$$d_{i_1} \geq k_{i_1} \lor d_{i_2} \geq k_{i_2} \lor \cdots \lor d_{i_r} \geq k_{i_r},$$

where all $i_j$ and $k_{i_j}$ are integers, with $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ and $1 \leq k_{i_1} \leq k_{i_2} \leq \cdots \leq k_{i_r} \leq n$. Given an $n$-sequence $\pi = (k_1 \leq k_2 \leq \cdots \leq k_n)$, let $C(\pi)$ denote the Chvátal-type condition:

$$d_1 \geq k_1 + 1 \lor d_2 \geq k_2 + 1 \lor \cdots \lor d_n \geq k_n + 1.$$ 

Intuitively, $C(\pi)$ is the weakest condition that 'blocks' $\pi$. For instance, if $\pi = 2^23^35$, then $C(\pi)$ is

$$d_1 \geq 3 \lor d_2 \geq 3 \lor d_3 \geq 4 \lor d_4 \geq 4 \lor d_5 \geq 4 \lor d_6 \geq 6. \quad (1)$$

Since $n$-sequences are assumed to be nondecreasing, $d_1 \geq 3$ implies $d_2 \geq 3$, etc. Also, we cannot have $d_i \geq n$, so the condition $d_6 \geq 6$ is redundant. Hence (1) can be simplified to the equivalent Chvátal-type condition

$$d_2 \geq 3 \lor d_5 \geq 4, \quad (2)$$

and we use $(1) \equiv (2)$ to denote this equivalence.

Conversely, given a Chvátal-type condition $c$, let $\Pi(c)$ denote the minimal $n$-sequence that majorizes all sequences which violate $c$ ($\Pi(c)$ might not be graphical). So if $c$ is the condition in (2) and $n = 6$, then $\Pi(c)$ is $2^23^35$. Of course, $\Pi(c)$ itself violates $c$. Note that $C$ and $\Pi$ are inverses: For any Chvátal-type condition $c$ we have $C(\Pi(c)) \equiv c$, and for any $n$-sequence $\pi$ we have $\Pi(C(\pi)) = \pi$.

Given a graph property $P$, we call a Chvátal-type degree condition $c$ $P$-weakly-optimal if any sequence $\pi$ (not necessarily graphical) which does not satisfy $c$ is majorized by a degree sequence which is not forcibly $P$. In particular, each of the $\left\lfloor \frac{1}{2} (n - 1) \right\rfloor$ conditions in Chvátal’s hamiltonian theorem is weakly optimal.

Next consider the poset whose elements are the graphical sequences of length $n$, with the majorization relation $\pi \leq \pi'$ as the partial order relation. We call this poset the $n$-degree-poset. Posets of integer sequences with a different order relation were previously used by Aigner & Triesch [1] in their work on graphical sequences.

Given a graph property $P$, consider the set of $n$-vertex graphs without property $P$ which are edge-maximal in this regard. The degree sequences of these edge-maximal,
non-$P$ graphs induce a subposet of the $n$-degree-poset, called the $P$-subposet. We refer to the maximal elements of this $P$-subposet as sinks, and denote their number by $s(n,P)$.

We first prove the following lemma.

**Lemma 3.1.** Let $P$ be a graph property. If a sink $\pi$ of the $P$-subposet violates a $P$-weakly-optimal Chvátal-type condition $c$, then $c \cong C(\pi)$.

**Proof:** Since $\pi$ violates $c$, $\pi \leq \Pi(c)$. Since $\Pi(c)$ violates $c$, and $c$ is $P$-weakly-optimal, there is a sequence $\pi' \geq \Pi(c)$ such that $\pi'$ has a non-$P$ realization. But $\pi' \leq \pi''$ for some sink $\pi''$, giving $\pi \leq \Pi(c) \leq \pi' \leq \pi''$. Since distinct sinks are incomparable, $\pi = \pi''$. This implies $\Pi(c) = \pi$, and thus $c \cong C(\Pi(c)) \cong C(\pi)$. ■

**Theorem 3.2.** Let $P$ be a graph property. Then any $P$-theorem for $n$-sequences whose hypothesis consists solely of $P$-weakly-optimal Chvátal-type conditions must contain at least $s(n,P)$ such conditions.

**Proof:** Consider a $P$-theorem whose hypothesis consists solely of $P$-weakly-optimal Chvátal-type conditions. By Lemma 3.1, a sink $\pi$ satisfies every Chvátal-type condition besides $C(\pi)$. So the theorem must include all the Chvátal-type conditions $C(\pi)$, as $\pi$ ranges over the $s(n,P)$ sinks. ■

On the other hand, it is easy to see that if we take the collection of Chvátal-type conditions $C(\pi)$ for all sinks $\pi$ in the $P$-subposet, then this gives a best monotone $P$-theorem.

We do not have a comparable result for $P$-theorems if we do not require the conditions to be $P$-weakly-optimal, let alone if we consider conditions that are not of Chvátal-type. On the other hand, all results we have discussed so far, and most of the forcibly $P$-theorems we know in the literature, involve only $P$-weakly-optimal Chvátal-type degree conditions.

### 4 Best Monotone $t$-Tough Theorems for $t \leq 1$

Using the terminology from Section 3, it follows that Theorem 2.1 gives, for $t \geq 1$, a best monotone $t$-tough theorem using a linear number (in $n$) of weakly optimal Chvátal-type conditions. On the other hand, we now show that for any integer $k \geq 1$, a best monotone $1/k$-tough theorem for $n$-sequences requires at least $f(k) \cdot n$ weakly optimal Chvátal-type conditions, where $f(k)$ grows superpolynomially as $k \to \infty$. In view of Theorem 3.2, to prove this assertion it suffices to prove the following lemma.
Lemma 4.1. Let \( k \geq 2 \) be an integer, and let \( n = m(k + 1) \) for some integer \( m \geq 9 \). Then the number of \((1/k\text{-tough})\)-subposet sinks in the \( n\)-degree-subposet is at least \( \frac{p(k-1)}{5(k+1)}n \), where \( p \) denotes the integer partition function.

Recall that the integer partition function \( p(r) \) counts the number of ways a positive integer \( r \) can be written as a sum of positive integers. Since \( p(r) \sim \frac{1}{4r\sqrt{3}}e^{\pi\sqrt{2r/3}} \) as \( r \to \infty \) [5], \( f(k) = \frac{p(k-1)}{5(k+1)} \) grows superpolynomially as \( k \to \infty \).

Proof of Lemma 4.1: Consider the collection \( \mathcal{C} \) of all connected graphs on \( n \) vertices which are edge-maximally not-\((1/k\text{-tough})\). Each \( G \in \mathcal{C} \) has the form \( G = K_j + (K_{c_0} \cup \cdots \cup K_{c_{kj+1}}) \), where \( j < n/(k+1) = m \), so that \( 1 \leq j \leq m-1 \), and \( c_1 + \cdots + c_{kj+1} \) is a partition of \( n-j \). Assuming \( c_1 \leq \cdots \leq c_{kj+1} \), the degree sequence of \( G \) becomes \( \pi \doteq (c_1 + j - 1)^{c_1} \cdots (c_{kj+1} + j - 1)^{c_{kj+1}}(n-1)^j \). Note that \( \pi \) cannot be majorized by the degrees of any disconnected graph on \( n \) vertices, since a disconnected graph has no vertex of degree \( n-1 \). By a complete degree of a degree sequence we mean an entry in the sequence equal to \( n-1 \).

Partition the degree sequences of the graphs in \( \mathcal{C} \) into \( m-1 \) groups, where the sequences in the \( j \)th group, \( 1 \leq j \leq m-1 \), are precisely those containing \( j \) complete degrees. We establish two basic properties of the \( j \)th group.

Claim 1. There are exactly \( p_{kj+1}((k+1)(m-j)-1) \) sequences in the \( j \)th group.

Here \( p_{\ell}(r) \) denotes the number of partitions of integer \( r \) into at most \( \ell \) parts, or equivalently the number of partitions of \( r \) with largest part at most \( \ell \).

Proof of Claim 1: Each sequence in the \( j \)th group corresponds uniquely to a set of \( kj+1 \) component sizes which sum to \( n-j \). If we subtract 1 from each of those component sizes, we obtain a corresponding collection of \( kj+1 \) integers (some possibly 0) which sum to \( n-j-(kj+1) = (k+1)(m-j)-1 \), and which therefore form a partition of \((k+1)(m-j)-1\) into at most \( kj+1 \) parts. \( \square \)

Claim 2. No sequence in the \( j \)th group majorizes another sequence in the \( j \)th group.

Proof: Suppose the sequences \( \pi \doteq (c_1 + j - 1)^{c_1} \cdots (c_{kj+1} + j - 1)^{c_{kj+1}}(n-1)^j \) and \( \pi' \doteq (c'_1 + j - 1)^{c'_1} \cdots (c'_{kj+1} + j - 1)^{c'_{kj+1}}(n-1)^j \) are in the \( j \)th group, with \( \pi \geq \pi' \). Deleting the \( j \) complete degrees from each sequence gives sequences \( \sigma \doteq (c_1 - 1)^{c_1} \cdots (c_{kj+1} - 1)^{c_{kj+1}} \) and \( \sigma' \doteq (c'_1 - 1)^{c'_1} \cdots (c'_{kj+1} - 1)^{c'_{kj+1}} \), with \( \sigma \geq \sigma' \).

Let \( m \) be the smallest index with \( c_m \neq c'_m \); since \( \sigma \geq \sigma' \), we have \( c_m > c'_m \). In particular, \( c_1 + \cdots + c_m > c'_1 + \cdots + c'_m \). But \( c_1 + \cdots + c_{kj+1} = c'_1 + \cdots + c'_{kj+1} = n-j \), and so there exists a smallest index \( \ell > m \) with \( c_1 + \cdots + c_{\ell} \leq c'_1 + \cdots + c'_{\ell} \). In particular, \( c_\ell < c'_\ell \). Since \( c'_1 + \cdots + c'_{\ell-1} < c_1 + \cdots + c_{\ell-1} < c_1 + \cdots + c_{\ell} \leq c_1 + \cdots + c'_\ell \),...
we have \( d_{c_1 + \ldots + c_\ell} = c_\ell - 1 < c'_\ell - 1 = d'_{c_1 + \ldots + c_\ell} \), and thus \( \sigma \not\prec \sigma' \), a contradiction. \( \square \)

Since \( K_j + (K_{c_1} \cup \ldots \cup K_{c_{k_j+1}}) \) has \( n \) vertices, \( K_{c_{k_j+1}} \) has at most \( n - j - kj \) vertices. This means the largest possible noncomplete degree in a sequence in the \( j \text{th} \) group is \( j + (n - j - kj - 1) = n - kj - 1 \). Using this observation we can prove the following.

**Claim 3.** If a sequence \( \pi = \ldots d^{d-j+1}(n-1)^i \) in the \( j \text{th} \) group has largest noncomplete degree \( d \geq n - k(j+1) \), then \( \pi \) is not majorized by any sequence in the \( i \text{th} \) group, for \( i \geq j + 1 \).

In particular, such a \( \pi \) is a sink, since \( \pi \) is certainly not majorized by another sequence in the \( j \text{th} \) group by Claim 2, nor by a sequence in groups \( 1, 2, \ldots, j - 1 \), since any such sequence has fewer than \( j \) complete degrees.

**Proof of Claim 3:** If \( d \geq n - k(j+1) \), then the \( d+1 \) largest degrees \( d^{d-j+1}(n-1)^i \) in \( \pi \) could be majorized only by complete degrees in a sequence in group \( i \geq j + 1 \), since the largest noncomplete degree in any sequence in group \( i \) is at most \( n - ki - 1 < n - k(j+1) \). There are only \( i \leq m - 1 \) complete degrees in a sequence in group \( i \). On the other hand, since \( j + 1 \leq i < m \), we have \( d + 1 \geq n - k(j+1) + 1 > m(k+1) - km + 1 = m + 1 > m - 1 \), a contradiction. \( \square \)

So by Claim 3, the sequences \( \pi \) in the \( j \text{th} \) group which could possibly be nonsinks (i.e., majorized by a sequence in group \( i \), for some \( i \geq j + 1 \)), must have largest noncomplete degree at most \( n - k(j+1) - 1 \). So in a graph \( G \in C \), \( G = K_j + (K_{c_1} \cup \ldots \cup K_{c_{k_j+1}}) \), which realizes a nonsink \( \pi \), each of the \( K_{c_i} \)'s must have order at most \( (n - k(j+1) - 1) - j + 1 = (k+1)(m-j) - k \). Subtracting 1 from the order of each of these components gives a sequence of \( kj + 1 \) integers (some possibly 0) which sum to \( (n-j)-(kj+1) = (k+1)(m-j) - 1 \), and which have largest part at most \( (k+1)(m-j) - k - 1 = (k+1)(m-j) - 1 \). Thus there are exactly \( p_{(k+1)(m-j-1)}((k+1)(m-j) - 1)) \) such sequences, and so there are at most this many nonsinks in the \( j \text{th} \) group. Setting \( N(j) = (k+1)(m-j) - 1 \), so that \( (k+1)(m-j-1) = N(j) - k \), this becomes at most \( p_{N(j)-k}(N(j)) \) nonsinks in the \( j \text{th} \) group of sequences.

But by Claim 1, there are exactly \( p_{kj+1}(N(j)) \) sequences in group \( j \), and so the number of sinks in the \( j \text{th} \) group is at least \( p_{kj+1}(N(j)) - p_{N(j)-k}(N(j)) \).

Note that \( p_{kj+1}(N(j)) \) reduces to \( p(N(j)) \) if \( kj+1 \geq N(j) \). However, \( kj+1 \geq N(j) \) is equivalent to \( j \geq \frac{(k+1)m - 2}{2k+1} \). Since \( k \geq 2 \), the inequality \( j \geq \frac{(k+1)m - 2}{2k+1} \) holds if \( j \geq \frac{3}{5}m \). Thus \( p_{kj+1}(N(j)) = p(N(j)) \) holds for \( j \geq \frac{3}{5}m \).

On the other hand, for \( j \leq m - 2 \) we can show the following.
Claim 4. If \( j \leq m - 2 \), then
\[
p(N(j)) - p_{N(j)-k}(N(j)) = 1 + p(1) + \cdots + p(k-1) \geq p(k-1).
\]

Proof: Note that if \( j \leq m - 2 \), then \( k < \frac{1}{2} N(j) \). The left side of the equality in the claim counts partitions of \( N(j) \) with largest part at least \( N(j) - (k-1) \). The right side counts the same according to the exact size \( N(j) - \ell \), \( 0 \leq \ell \leq k-1 \), of the largest part in the partition, using that the largest part is unique since \( N(j) - \ell \geq N(j) - (k-1) > \frac{1}{2} N(j) \). \( \square \)

Completing the proof of Lemma 4.1, we find that the number of sinks in the \((1/k\text{-tough})\)-subposet of the \( n \)-degree-poset is at least
\[
\sum_{j=\lceil 3m/5 \rceil}^{m-2} \left[ p_{kj+1}(N(j)) - p_{N(j)-k}(N(j)) \right] = \sum_{j=\lceil 3m/5 \rceil}^{m-2} \left[ p(N(j)) - p_{N(j)-k}(N(j)) \right]
\geq \sum_{j=\lceil 3m/5 \rceil}^{m-2} p(k-1) \geq \left( \frac{2}{5} m - \frac{9}{5} \right) p(k-1)
= \left( \frac{2n}{5(k+1)} - \frac{9}{5} \right) p(k-1) \geq \frac{n}{5(k+1)} p(k-1),
\]
as asserted, since \( n = m(k+1) \geq 9(k+1) \) implies \( \frac{2n}{5(k+1)} - \frac{9}{5} \geq \frac{n}{5(k+1)} \). \( \square \)

Combining Lemma 4.1 with Theorem 3.2 gives the promised superpolynomial growth in the number of weakly optimal Chvátal-type conditions for \( 1/k \)-toughness.

Theorem 4.2. Let \( k \geq 2 \) be an integer, and let \( n = m(k+1) \) for some integer \( m \geq 9 \). Then a best monotone \( 1/k \text{-tough} \) theorem for \( n \)-sequences whose degree conditions consist solely of weakly optimal Chvátal-type conditions requires at least
\[
\frac{p(k-1)n}{5(k+1)}
\]
such conditions, where \( p(r) \) is the integer partition function.

5 A Simple \( t \)-Tough Theorem

The superpolynomial complexity as \( k \to \infty \) of a best monotone \( 1/k \text{-tough} \) theorem suggests the desirability of finding simple \( t \)-tough theorems, when \( t < 1 \). We give such a theorem below. It will again be convenient to assume at first that \( t = 1/k \), for some integer \( k \geq 1 \). Note that the conditions in the theorem are still Chvátal-type conditions.
Lemma 5.1. Let $k \geq 1$ be an integer, $n \geq k + 2$, and $\pi = (d_1 \leq \cdots \leq d_n)$ a graphical sequence. If

(i) $d_i \geq i - k + 2$ or $d_{n-i+k-1} \geq n-i$, for $k \leq i < \frac{1}{2}(n+k-1)$, and

(ii) $d_i \geq i$ or $d_n \geq n-i$, for $1 \leq i \leq \frac{1}{2}n$,

then $\pi$ is forcibly $1/k$-tough.

Proof of Lemma 5.1: Suppose $\pi$ has a realization $G$ which is not $1/k$-tough. By (ii) and Theorem 1.2, $G$ is connected. So we may assume (by adding edges if necessary) that there exists $X \subseteq V(G)$, with $|X| \geq 1$, such that $G = K_x + (K_{a_1} \cup K_{a_2} \cup \cdots \cup K_{a_{kx+1}})$, where $1 \leq a_1 \leq a_2 \leq \cdots \leq a_{kx+1}$.

Set $i = x + k - 2 + a_{kx}$.

Claim 1. $k \leq i < \frac{1}{2}(n+k-1)$

Proof: The fact that $i \geq k$ follows immediately from the definition of $i$. Since $kx - x - k + 1 = (k-1)(x-1) \geq 0$, we have

$$kx - 1 \geq x + k - 2. \quad (3)$$

This leads to

$$n = x + \sum_{j=1}^{kx-1} a_j + a_{kx} + a_{kx+1} \geq x + kx - 1 + 2a_{kx} \geq 2x + k - 2 + 2a_{kx} = 2i - k + 2,$$

which is equivalent to $i < \frac{1}{2}(n+k-1)$. \hfill \Box

Claim 2. $d_i \leq i - k + 1$.

Proof: From (3) we get

$$i = x + k - 2 + a_{kx} \leq kx - 1 + a_{kx} \leq \sum_{j=1}^{kx} a_j. \quad (4)$$

This gives $d_i \leq x + (a_{kx} - 1) = i - k + 1$. \hfill \Box

Claim 3. $d_{n-i+k-1} < n-i$.

Proof: We have $n-i+k-1 = n - x - a_{kx} + 1 \leq \sum_{j=1}^{kx+1} a_j$. Thus, using the bound (4) for $i$,

$$d_{n-i+k-1} \leq x + a_{kx+1} - 1 < n - \sum_{j=1}^{kx} a_j \leq n - i.$$ \hfill \Box
Claims 1, 2 and 3 together contradict condition (i), completing the proof of the lemma.

We can extend Lemma 5.1 to arbitrary $t \leq 1$ by letting $k = \lfloor 1/t \rfloor$.

**Theorem 5.2.** Let $t \leq 1$, $n \geq \lfloor 1/t \rfloor + 2$, and $\pi = (d_1 \leq \cdots \leq d_n)$ a graphical sequence. If

(i) $d_i \geq i - \lfloor 1/t \rfloor + 2$ or $d_{n-i+\lfloor 1/t \rfloor -1} \geq n - i$, for $[1/t] \leq i < \frac{1}{2}(n + \lfloor 1/t \rfloor - 1)$, and

(ii) $d_i \geq i$ or $d_n \geq n - i$, for $1 \leq i \leq \frac{1}{2}n$,

then $\pi$ is forcibly $t$-tough.

**Proof:** Set $k = \lfloor 1/t \rfloor \geq 1$. If $\pi$ satisfies conditions (i), (ii) in Theorem 5.2, then $\pi$ satisfies conditions (i), (ii) in Lemma 5.1, and so is forcibly $1/k$-tough. But $k = \lfloor 1/t \rfloor \leq 1/t$ means $1/k \geq t$, and so $\pi$ is forcibly $t$-tough.

In summary, if $\frac{1}{k+1} < t \leq \frac{1}{k}$ for some integer $k \geq 1$, then Theorem 5.2 declares $\pi$ forcibly $t$-tough precisely if Lemma 5.1 declares $\pi$ forcibly $1/k$-tough.

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**References**


