Toughness and Vertex Degrees

D. Bauer
Department of Mathematical Sciences
Stevens Institute of Technology
Hoboken, NJ 07030, U.S.A.

H.J. Broersma*
School of Engineering
and Computing Sciences
Durham University
South Road, Durham DH1 3LE, U.K.

J. van den Heuvel
Department of Mathematics
London School of Economics
Houghton Street, London WC2A 2AE, U.K.

N. Kahl
Department of Mathematics
and Computer Science
Seton Hall University
South Orange, NJ 07079, U.S.A.

E. Schmeichel
Department of Mathematics
San José State University
San José, CA 95192, U.S.A.

Abstract

We study theorems giving sufficient conditions on the vertex degrees of a graph G to guarantee G is t-tough. We first give a best monotone theorem when $t \geq 1$, but then show that for any integer $k \geq 1$, a best monotone theorem for $t = \frac{1}{k} \leq 1$ requires at least $f(k) \cdot |V(G)|$ nonredundant conditions, where $f(k)$ grows superpolynomially as $k \to \infty$. When $t < 1$, we give an additional, simple theorem for G to be t-tough, in terms of its vertex degrees.

1 Introduction

We consider only simple graphs without loops or multiple edges. Our terminology and notation will be standard except as indicated, and a good reference for any undefined terms or notation is [7]. For two graphs G, H on disjoint vertex sets, we

* Supported by EPSRC Grant EP/F064551/1.

Current address: Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands.

denote their *union* by $G \cup H$. The *join* $G + H$ of G and H is the graph formed from $G \cup H$ by adding all edges between $V(G)$ and $V(H)$.

For a positive integer n, an *n-sequence* (or just a *sequence*) is an integer sequence $\pi = (d_1, d_2, \ldots, d_n)$, with $0 \le d_j \le n-1$ for all j. In contrast to [7], we will usually write the sequence in nondecreasing order (and may make this explicit by writing $\pi = (d_1 \le \cdots \le d_n)$). We will employ the standard abbreviated notation for sequences, e.g., $(4,4,4,4,4,5,5,6)$ will be denoted $4^5\, 5^2\, 6^1$. If $\pi = (d_1, \ldots, d_n)$ and $\pi' = (d'_1, \ldots, d'_n)$ are two n-sequences, we say π' *majorizes* π, denoted $\pi' \ge \pi$, if $d'_j \ge d_j$ for all j.

A *degree sequence* of a graph is any sequence $\pi = (d_1, d_2, \ldots, d_n)$ consisting of the vertex degrees of the graph. A sequence π is *graphical* if there exists a graph G having π as one of its degree sequences, in which case we call G a *realization* of π. If P is a graph property (e.g., hamiltonian, k-connected), we call a graphical sequence π *forcibly P* if every realization of π has property P.

Historically, the degree sequence of a graph has been used to provide sufficient conditions for a graph to have certain properties, such as hamiltonicity or k-connectivity. In particular, sufficient conditions for π to be forcibly hamiltonian were given by several authors, culminating in the following theorem of Chvátal [4].

Theorem 1.1 ([4]). *Let $\pi = (d_1 \le \cdots \le d_n)$ be a graphical sequence, with $n \ge 3$. If $d_i \le i < \frac{1}{2}n$ implies $d_{n-i} \ge n-i$, then π is forcibly hamiltonian.*

Unlike its predecessors, Chvátal's theorem has the property that if it does not guarantee that π is forcibly hamiltonian because the condition fails for some $i < \frac{1}{2}n$, then π is majorized by $\pi' = i^i\,(n-i-1)^{n-2i}\,(n-1)^i$, which has a unique nonhamiltonian realization $K_i + (\overline{K_i} \cup K_{n-2i})$. As we will see below, this implies that Chvátal's theorem is the strongest of an entire class of theorems giving sufficient degree conditions for π to be forcibly hamiltonian.

Sufficient conditions for π to be forcibly k-connected were given by several authors, culminating in the following theorem of Bondy [3] (though the form in which we present it is due to Boesch [2]).

Theorem 1.2 ([2, 3]). *Let $\pi = (d_1 \le \cdots \le d_n)$ be a graphical sequence with $n \ge 2$, and let $1 \le k \le n-1$. If $d_i \le i+k-2$ implies $d_{n-k+1} \ge n-i$, for $1 \le i \le \frac{1}{2}(n-k+1)$, then π is forcibly k-connected.*

Boesch [2] also observed that Theorem 1.2 is the strongest theorem giving sufficient degree conditions for π to be forcibly k-connected, in exactly the same sense as Theorem 1.1.

Let $\omega(G)$ denote the number of components of a graph G. For $t \ge 0$, we call G *t-tough* if $t \cdot \omega(G - X) \le |X|$, for every $X \subseteq V(G)$ with $\omega(G - X) > 1$. The *toughness* of G, denoted $\tau(G)$, is the maximum $t \geq 0$ for which G is t-tough (taking $\tau(K_n) = n-1$, for all $n \geq 1$). So if G is not complete, then

$$\tau(G) = \min\Big\{\frac{|X|}{\omega(G-X)} \;\Big|\; X \subseteq V(G) \text{ is a cutset of } G\Big\}.$$

In this paper we consider forcibly t-tough theorems, for any $t \geq 0$. When trying to formulate and prove this type of theorem, we encountered very different behavior in the number of conditions required for a best possible theorem for the cases $t \geq 1$ and $t < 1$. In order to describe this behavior precisely, we need to say what we mean by a 'condition' and by a 'best possible theorem'.

First note that the conditions in Theorems 1.1 can be written in the form:

$$d_i \geq i+1 \;\text{ or }\; d_{n-i} \geq n-i, \;\text{ for } i = 1, \ldots, \left\lfloor \tfrac{1}{2}(n-1) \right\rfloor,$$

and the conditions in Theorem 1.2 can be written in a similar way. We will use the term 'Chvátal-type conditions' for such conditions. Formally, a *Chvátal-type condition* for n-sequences $(d_1 \leq d_2 \leq \cdots \leq d_n)$ is a condition of the form

$$d_{i_1} \geq k_{i_1} \;\vee\; d_{i_2} \geq k_{i_2} \;\vee\; \ldots \;\vee\; d_{i_r} \geq k_{i_r},$$

where all i_j and k_{i_j} are integers, with $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ and $1 \leq k_{i_1} \leq k_{i_2} \leq \cdots \leq k_{i_r} \leq n$.

A graph property P is called *increasing* if whenever a graph G has P, so has every edge-augmented supergraph of G. In particular, "hamiltonian", "k-connected" and "t-tough" are all increasing graph properties. In this paper, the term "graph property" will always mean an increasing graph property.

Given a graph property P, consider a theorem T which declares certain degree sequences to be forcibly P, rendering no decision on the remaining degree sequences. We call such a theorem T a *forcibly P-theorem* (or just a *P-theorem*, for brevity). Thus Theorem 1.1 would be a forcibly hamiltonian theorem. We call a P-theorem T *monotone* if, for any two degree sequences π, π', whenever T declares π forcibly P and $\pi' \geq \pi$, then T declares π' forcibly P. We call a P-theorem T *optimal* if whenever T does not declare a degree sequence π forcibly P, then π is not forcibly P; T is *weakly optimal* if for any sequence π (not necessarily graphical) which T does not declare forcibly P, π is majorized by a degree sequence which is not forcibly P.

A P-theorem which is both monotone and weakly optimal is a best monotone P-theorem, in the following sense.

Theorem 1.3. *Let T, T_0 be monotone P-theorems, with T_0 weakly optimal. If T declares a degree sequence π to be forcibly P, then so does T_0.*

Proof of Theorem 1.3: Suppose to the contrary that there exists a degree sequence π so that T declares π forcibly P, but T_0 does not. Since T_0 is weakly optimal, there exists a degree sequence $\pi' \geq \pi$ which is not forcibly P. This means that also T will not declare π' forcibly P. But if T declares π forcibly P, $\pi' \geq \pi$, and T does not declare π' forcibly P, then T is not monotone, a contradiction. ■

If T_0 is Chvátal's hamiltonian theorem (Theorem 1.1), then T_0 is clearly monotone, and we noted above that T_0 is weakly optimal. So by Theorem 1.3, Chvátal's theorem is a best monotone hamiltonian theorem.

Our goal in this paper is to consider forcibly t-tough theorems, for any $t \geq 0$. In Section 2 we first give a best monotone t-tough theorem for n-sequences, requiring at most $\lfloor \frac{1}{2}n \rfloor$ Chvátal-type conditions, for any $t \geq 1$. In contrast to this, in Sections 3 and 4 we show that for any integer $k \geq 1$, a best monotone $1/k$-tough theorem contains at least $f(k) \cdot n$ nonredundant Chvátal-type conditions, where $f(k)$ grows superpolynomially as $k \to \infty$. A similar superpolynomial growth in the complexity of the best monotone k-edge-connected theorem in terms of k was previously noted by Kriesell [6].

This superpolynomial complexity of a best monotone $1/k$-tough theorem suggests the desirability of finding more reasonable t-tough theorems, when $t < 1$. In Section 5 we give one such theorem. This theorem is a monotone, though not best monotone, t-tough theorem which is valid for any $t \leq 1$.

2 A Best Monotone t-Tough Theorem for $t \geq 1$

We first give a best monotone t-tough theorem for $t \geq 1$.

Theorem 2.1. *Let $t \geq 1$, $n \geq \lceil t \rceil + 2$, and let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence. If*

$$(*t) \qquad d_{\lfloor i/t \rfloor} \geq i+1 \quad \text{or} \quad d_{n-i} \geq n - \lfloor i/t \rfloor, \quad \text{for } t \leq i < \frac{tn}{(t+1)},$$

then π is forcibly t-tough.

Clearly, property $(*t)$ in Theorem 2.1 is monotone. Furthermore, if π does not satisfy $(*t)$ for some i with $t \leq i < tn/(t+1)$, then π is majorized by $\pi' = i^{\lfloor i/t \rfloor} (n - \lfloor i/t \rfloor - 1)^{n-i-\lfloor i/t \rfloor} (n-1)^i$, which has the non-$t$-tough realization $K_i + \left(\overline{K_{\lfloor i/t \rfloor}} \cup K_{n-i-\lfloor i/t \rfloor}\right)$. Thus $(*t)$ in Theorem 2.1 is also weakly optimal, and so Theorem 2.1 is best monotone by Theorem 1.3. Finally, note that when $t = 1$, $(*t)$ reduces to Chvátal's hamiltonian condition in Theorem 1.1.

Proof of Theorem 2.1: Suppose π satisfies $(*t)$ for some $t \geq 1$ and $n \geq \lceil t \rceil + 2$, but π has a realization G which is not t-tough. Then there exists a set $X \subseteq V(G)$ that is maximal with respect to $\omega(G - X) \geq 2$ and $\dfrac{|X|}{\omega(G-X)} < t$. Let $x \doteq |X|$

and $w \doteq \omega(G - X)$, so that $w \geq \lfloor x/t \rfloor + 1$. Also, let $H_1, H_2, \ldots, H_w$ denote the components of $G - X$, with $|H_1| \geq |H_2| \geq \cdots \geq |H_w|$, and let $h_j \doteq |H_j|$ for $j = 1, \ldots, w$. By adding edges (if needed) to G, we may assume $\langle X \rangle$ is complete, and each $\langle H_j \rangle$ is complete and completely joined to X.

Set $i \doteq x + h_2 - 1$.

Claim 1. $i \geq t$.

Proof: It is enough to show that $x \geq t$. Assume instead that $x < t$. Define $X' \doteq X \cup \{v\}$, with $v \in H_1$. If $h_1 \geq 2$, then

$$\frac{|X'|}{\omega(G - X')} = \frac{x+1}{\omega(G - X)} < \frac{t+1}{2} \leq t,$$

which contradicts the maximality of X. Similarly, if $h_1 = 1$ and $w \geq 3$, then

$$\frac{|X'|}{\omega(G - X')} = \frac{x+1}{\omega(G - X) - 1} < \frac{t+1}{2} \leq t,$$

also a contradiction. Finally, if $h_1 = 1$ and $w = 2$, then G is the graph $K_{n-2} + \overline{K_2}$ with $n - 2 = x < t$, contradicting $n \geq \lceil t \rceil + 2$. □

Claim 2. $i < \dfrac{tn}{t+1}$

Proof: Note that $n = x + h_1 + h_2 + \cdots + h_w \geq x + 2h_2 + w - 2$. Since $x < tw$, we obtain

$$\begin{aligned}(t+1)i &= (t+1)(x + h_2 - 1) = t(x + h_2 - 1) + x + h_2 - 1 \\ &< t(x + h_2 - 1) + tw + t(h_2 - 1) \leq tn. \end{aligned} \qquad \square$$

By the claims we have $t \leq i < \dfrac{tn}{t+1}$. Next note that

$$\left\lfloor \frac{i}{t} \right\rfloor = \left\lfloor \frac{x + h_2 - 1}{t} \right\rfloor \leq \left\lfloor \frac{x}{t} \right\rfloor + h_2 - 1 \leq w + h_2 - 2 \leq \sum_{j=2}^{w} h_j = n - x - h_1,$$

so

$$d_{\lfloor i/t \rfloor} \leq d_{=n-x-h_1} = x + h_2 - 1 = i.$$

However, we also have

$$\begin{aligned} d_{n-i} \leq d_{n-x} &= x + h_1 - 1 = n - h_2 - (h_3 + \cdots + h_w) - 1 \leq n - (w + h_2 - 1) \\ &< n - \left(\frac{x}{t} + h_2 - 1\right) \leq n - \frac{x + h_2 - 1}{t} = n - i/t \leq n - \lfloor i/t \rfloor, \end{aligned}$$

contradicting $(*t)$. ■

3 The Number of Chvátal-Type Conditions in Best Monotone Theorems

In this section we provide a theory that allows us to lower bound the number of degree sequence conditions required in a best monotone P-theorem.

Recall that a *Chvátal-type condition* for n-sequences $(d_1 \leq d_2 \leq \cdots \leq d_n)$ is a condition of the form

$$d_{i_1} \geq k_{i_1} \ \vee \ d_{i_2} \geq k_{i_2} \ \vee \ \ldots \ \vee \ d_{i_r} \geq k_{i_r},$$

where all i_j and k_{i_j} are integers, with $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ and $1 \leq k_{i_1} \leq k_{i_2} \leq \cdots \leq k_{i_r} \leq n$. Given an n-sequence $\pi = (k_1 \leq k_2 \leq \cdots \leq k_n)$, let $C(\pi)$ denote the Chvátal-type condition:

$$d_1 \geq k_1 + 1 \ \vee \ d_2 \geq k_2 + 1 \ \vee \ \ldots \ \vee \ d_n \geq k_n + 1.$$

Intuitively, $C(\pi)$ is the weakest condition that 'blocks' π. For instance, if $\pi = 2^2 3^3 5$, then $C(\pi)$ is

$$d_1 \geq 3 \ \vee \ d_2 \geq 3 \ \vee \ d_3 \geq 4 \ \vee \ d_4 \geq 4 \ \vee \ d_5 \geq 4 \ \vee \ d_6 \geq 6. \tag{1}$$

Since n-sequences are assumed to be nondecreasing, $d_1 \geq 3$ implies $d_2 \geq 3$, etc. Also, we cannot have $d_i \geq n$, so the condition $d_6 \geq 6$ is redundant. Hence (1) can be simplified to the equivalent Chvátal-type condition

$$d_2 \geq 3 \ \vee \ d_5 \geq 4, \tag{2}$$

and we use (1) $\cong$ (2) to denote this equivalence.

Conversely, given a Chvátal-type condition c, let $\Pi(c)$ denote the minimal n-sequence that majorizes all sequences which violate c ($\Pi(c)$ might not be graphical). So if c is the condition in (2) and $n = 6$, then $\Pi(c)$ is $2^2 3^3 5$. Of course, $\Pi(c)$ itself violates c. Note that C and Π are inverses: For any Chvátal-type condition c we have $C(\Pi(c)) \cong c$, and for any n-sequence π we have $\Pi(C(\pi)) = \pi$.

Given a graph property P, we call a Chvátal-type degree condition c *P-weakly-optimal* if any sequence π (not necessarily graphical) which does not satisfy c is majorized by a degree sequence which is not forcibly P. In particular, each of the $\left\lfloor \frac{1}{2}(n-1) \right\rfloor$ conditions in Chvátal's hamiltonian theorem is weakly optimal.

Next consider the poset whose elements are the graphical sequences of length n, with the majorization relation $\pi \leq \pi'$ as the partial order relation. We call this poset the *n-degree-poset*. Posets of integer sequences with a different order relation were previously used by Aigner & Triesch [1] in their work on graphical sequences.

Given a graph property P, consider the set of n-vertex graphs without property P which are edge-maximal in this regard. The degree sequences of these edge-maximal, non-P graphs induce a subposet of the n-degree-poset, called the *P-subposet*. We refer to the maximal elements of this P-subposet as *sinks*, and denote their number by $s(n, P)$.

We first prove the following lemma.

Lemma 3.1. *Let P be a graph property. If a sink π of the P-subposet violates a P-weakly-optimal Chvátal-type condition c, then $c \cong C(\pi)$.*

Proof: Since π violates c, $\pi \leq \Pi(c)$. Since $\Pi(c)$ violates c, and c is P-weakly-optimal, there is a sequence $\pi' \geq \Pi(c)$ such that π' has a non-P realization. But $\pi' \leq \pi''$ for some sink π'', giving $\pi \leq \Pi(c) \leq \pi' \leq \pi''$. Since distinct sinks are incomparable, $\pi = \pi''$. This implies $\Pi(c) = \pi$, and thus $c \cong C(\Pi(c)) \cong C(\pi)$. ■

Theorem 3.2. *Let P be a graph property. Then any P-theorem for n-sequences whose hypothesis consists solely of P-weakly-optimal Chvátal-type conditions must contain at least $s(n, P)$ such conditions.*

Proof: Consider a P-theorem whose hypothesis consists solely of P-weakly-optimal Chvátal-type conditions. By Lemma 3.1, a sink π satisfies every Chvátal-type condition besides $C(\pi)$. So the theorem must include all the Chvátal-type conditions $C(\pi)$, as π ranges over the $s(n, P)$ sinks. ■

On the other hand, it is easy to see that if we take the collection of Chvátal-type conditions $C(\pi)$ for all sinks π in the P-subposet, then this gives a best monotone P-theorem.

We do not have a comparable result for P-theorems if we do not require the conditions to be P-weakly-optimal, let alone if we consider conditions that are not of Chvátal-type. On the other hand, all results we have discussed so far, and most of the forcibly P-theorems we know in the literature, involve only P-weakly-optimal Chvátal-type degree conditions.

4 Best Monotone t-Tough Theorems for $t \leq 1$

Using the terminology from Section 3, it follows that Theorem 2.1 gives, for $t \geq 1$, a best monotone t-tough theorem using a linear number (in n) of weakly optimal Chvátal-type conditions. On the other hand, we now show that for any integer $k \geq 1$, a best monotone $1/k$-tough theorem for n-sequences requires at least $f(k) \cdot n$ weakly optimal Chvátal-type conditions, where $f(k)$ grows superpolynomially as $k \to \infty$. In view of Theorem 3.2, to prove this assertion it suffices to prove the following lemma.

Lemma 4.1. *Let $k \geq 2$ be an integer, and let $n = m(k+1)$ for some integer $m \geq 9$. Then the number of (1/k-tough)-subposet sinks in the n-degree-subposet is at least $\dfrac{p(k-1)}{5(k+1)}n$, where p denotes the integer partition function.*

Recall that the integer partition function $p(r)$ counts the number of ways a positive integer r can be written as a sum of positive integers. Since $p(r) \sim \dfrac{1}{4r\sqrt{3}}e^{\pi\sqrt{2r/3}}$ as $r \to \infty$ [5], $f(k) = \dfrac{p(k-1)}{5(k+1)}$ grows superpolynomially as $k \to \infty$.

Proof of Lemma 4.1: Consider the collection $\mathcal{C}$ of all connected graphs on n vertices which are edge-maximally not-(1/k-tough). Each $G \in \mathcal{C}$ has the form $G = K_j + (K_{c_1} \cup \cdots \cup K_{c_{kj+1}})$, where $j < n/(k+1) = m$, so that $1 \leq j \leq m-1$, and $c_1 + \cdots + c_{kj+1}$ is a partition of $n-j$. Assuming $c_1 \leq \cdots \leq c_{kj+1}$, the degree sequence of G becomes $\pi \doteq (c_1+j-1)^{c_1} \ldots (c_{kj+1}+j-1)^{c_{kj+1}} (n-1)^j$. Note that π cannot be majorized by the degrees of any disconnected graph on n vertices, since a disconnected graph has no vertex of degree $n-1$. By a *complete degree* of a degree sequence we mean an entry in the sequence equal to $n-1$.

Partition the degree sequences of the graphs in $\mathcal{C}$ into $m-1$ groups, where the sequences in the j^{th} group, $1 \leq j \leq m-1$, are precisely those containing j complete degrees. We establish two basic properties of the j^{th} group.

Claim 1. *There are exactly $p_{kj+1}\big((k+1)(m-j)-1\big)$ sequences in the j^{th} group.*

Here $p_\ell(r)$ denotes the number of partitions of integer r into at most ℓ parts, or equivalently the number of partitions of r with largest part at most ℓ.

Proof of Claim 1: Each sequence in the j^{th} group corresponds uniquely to a set of $kj+1$ component sizes which sum to $n-j$. If we subtract 1 from each of those component sizes, we obtain a corresponding collection of $kj+1$ integers (some possibly 0) which sum to $n-j-(kj+1) = (k+1)(m-j)-1$, and which therefore form a partition of $(k+1)(m-j)-1$ into at most $kj+1$ parts. $\square$

Claim 2. *No sequence in the j^{th} group majorizes another sequence in the j^{th} group.*

Proof: Suppose the sequences $\pi \doteq (c_1+j-1)^{c_1} \ldots (c_{kj+1}+j-1)^{c_{kj+1}} (n-1)^j$ and $\pi' \doteq (c'_1+j-1)^{c'_1} \ldots (c'_{kj+1}+j-1)^{c'_{kj+1}} (n-1)^j$ are in the j^{th} group, with $\pi \geq \pi'$. Deleting the j complete degrees from each sequence gives sequences $\sigma \doteq (c_1-1)^{c_1} \ldots (c_{kj+1}-1)^{c_{kj+1}}$ and $\sigma' \doteq (c'_1-1)^{c'_1} \ldots (c'_{kj+1}-1)^{c'_{kj+1}}$, with $\sigma \geq \sigma'$.

Let m be the smallest index with $c_m \neq c'_m$; since $\sigma \geq \sigma'$, we have $c_m > c'_m$. In particular, $c_1+\cdots+c_m > c'_1+\cdots+c'_m$. But $c_1+\cdots+c_{kj+1} = c'_1+\cdots+c'_{kj+1} = n-j$, and so there exists a smallest index $\ell > m$ with $c_1+\cdots+c_\ell \leq c'_1+\cdots+c'_\ell$. In particular, $c_\ell < c'_\ell$. Since $c'_1+\cdots+c'_{\ell-1} < c_1+\cdots+c_{\ell-1} < c_1+\cdots+c_\ell \leq c_1+\cdots+c'_\ell$, we have $d_{c_1+\cdots+c_\ell} = c_\ell - 1 < c'_\ell - 1 = d'_{c_1+\cdots+c_\ell}$, and thus $\sigma \not\geq \sigma'$, a contradiction. $\square$

Since $K_j + (K_{c_1} \cup \cdots \cup K_{c_{kj+1}})$ has n vertices, $K_{c_{kj+1}}$ has at most $n - j - kj$ vertices. This means the largest possible noncomplete degree in a sequence in the j^{th} group is $j + (n - j - kj - 1) = n - kj - 1$. Using this observation we can prove the following.

Claim 3. *If a sequence $\pi = \cdots d^{d-j+1} (n-1)^j$ in the j^{th} group has largest noncomplete degree $d \geq n - k(j+1)$, then π is not majorized by any sequence in the i^{th} group, for $i \geq j+1$.*

In particular, such a π is a sink, since π is certainly not majorized by another sequence in the j^{th} group by Claim 2, nor by a sequence in groups $1, 2, \ldots, j-1$, since any such sequence has fewer than j complete degrees.

Proof of Claim 3: If $d \geq n - k(j+1)$, then the $d+1$ largest degrees $d^{d-j+1} (n-1)^j$ in π could be majorized only by complete degrees in a sequence in group $i \geq j+1$, since the largest noncomplete degree in any sequence in group i is at most $n - ki - 1 < n - k(j+1)$. There are only $i \leq m-1$ complete degrees in a sequence in group i. On the other hand, since $j+1 \leq i < m$, we have $d + 1 \geq n - k(j+1) + 1 > m(k+1) - km + 1 = m + 1 > m - 1$, a contradiction. $\square$

So by Claim 3, the sequences π in the j^{th} group which could possibly be nonsinks (i.e., majorized by a sequence in group i, for some $i \geq j+1$), must have largest noncomplete degree at most $n - k(j+1) - 1$. So in a graph $G \in \mathcal{C}$, $G = K_j + (K_{c_1} \cup \cdots \cup K_{c_{kj+1}})$, which realizes a nonsink π, each of the K_c's must have order at most $(n - k(j+1) - 1) - j + 1 = (k+1)(m-j) - k$. Subtracting 1 from the order of each of these components gives a sequence of $kj+1$ integers (some possibly 0) which sum to $(n-j) - (kj+1) = (k+1)(m-j) - 1$, and which have largest part at most $(k+1)(m-j) - k - 1 = (k+1)(m-j-1)$. Thus there are exactly $p_{(k+1)(m-j-1)}\big((k+1)(m-j)-1\big)$ such sequences, and so there are at most this many nonsinks in the j^{th} group. Setting $N(j) \doteq (k+1)(m-j) - 1$, so that $(k+1)(m-j-1) = N(j) - k$, this becomes at most $p_{N(j)-k}\big(N(j)\big)$ nonsinks in the j^{th} group of sequences.

But by Claim 1, there are exactly $p_{kj+1}\big(N(j)\big)$ sequences in group j, and so the number of sinks in the j^{th} group is at least $p_{kj+1}\big(N(j)\big) - p_{N(j)-k}\big(N(j)\big)$.

Note that $p_{kj+1}(N(j))$ reduces to $p(N(j))$ if $kj+1 \geq N(j)$. However, $kj+1 \geq N(j)$ is equivalent to $j \geq \dfrac{(k+1)m-2}{2k+1}$. Since $k \geq 2$, the inequality $j \geq \dfrac{(k+1)m-2}{2k+1}$ holds if $j \geq \frac{3}{5}m$. Thus $p_{kj+1}(N(j)) = p(N(j))$ holds for $j \geq \frac{3}{5}m$.

On the other hand, for $j \leq m-2$ we can show the following.

Claim 4. *If $j \le m-2$, then*

$$p\big(N(j)\big) - p_{N(j)-k}\big(N(j)\big) \;=\; 1 + p(1) + \cdots + p(k-1) \;\ge\; p(k-1).$$

Proof: Note that if $j \le m-2$, then $k < \frac{1}{2}N(j)$. The left side of the equality in the claim counts partitions of $N(j)$ with largest part at least $N(j)-(k-1)$. The right side counts the same according to the exact size $N(j)-\ell$, $0 \le \ell \le k-1$, of the largest part in the partition, using that the largest part is unique since $N(j)-\ell \ge N(j)-(k-1) > \frac{1}{2}N(j)$. □

Completing the proof of Lemma 4.1, we find that the number of sinks in the $(1/k$-tough)-subposet of the n-degree-poset is at least

$$\begin{aligned}\sum_{j=\lceil 3m/5\rceil}^{m-2} \big[p_{kj+1}\big(N(j)\big) - p_{N(j)-k}\big(N(j)\big)\big] \;&=\; \sum_{j=\lceil 3m/5\rceil}^{m-2} \big[p\big(N(j)\big) - p_{N(j)-k}\big(N(j)\big)\big] \\ &\ge\; \sum_{j=\lceil 3m/5\rceil}^{m-2} p(k-1) \;\ge\; \big(\tfrac{2}{5}m - \tfrac{9}{5}\big)p(k-1) \\ &=\; \Big(\frac{2n}{5(k+1)} - \frac{9}{5}\Big)p(k-1) \;\ge\; \frac{n}{5(k+1)}p(k-1),\end{aligned}$$

as asserted, since $n = m(k+1) \ge 9(k+1)$ implies $\dfrac{2n}{5(k+1)} - \dfrac{9}{5} \ge \dfrac{n}{5(k+1)}$. ■

Combining Lemma 4.1 with Theorem 3.2 gives the promised superpolynomial growth in the number of weakly optimal Chvátal-type conditions for $1/k$-toughness.

Theorem 4.2. *Let $k \ge 2$ be an integer, and let $n = m(k+1)$ for some integer $m \ge 9$. Then a best monotone $1/k$-tough theorem for n-sequences whose degree conditions consist solely of weakly optimal Chvátal-type conditions requires at least $\dfrac{p(k-1)n}{5(k+1)}$ such conditions, where $p(r)$ is the integer partition function.*

5 A Simple t-Tough Theorem

The superpolynomial complexity as $k \to \infty$ of a best monotone $1/k$-tough theorem suggests the desirability of finding simple t-tough theorems, when $t < 1$. We give such a theorem below. It will again be convenient to assume at first that $t = 1/k$, for some integer $k \ge 1$. Note that the conditions in the theorem are still Chvátal-type conditions.

Lemma 5.1. *Let $k \geq 1$ be an integer, $n \geq k+2$, and $\pi = (d_1 \leq \cdots \leq d_n)$ a graphical sequence. If*

(i) $d_i \geq i-k+2$ *or* $d_{n-i+k-1} \geq n-i$, *for* $k \leq i < \frac{1}{2}(n+k-1)$, *and*

(ii) $d_i \geq i$ *or* $d_n \geq n-i$, *for* $1 \leq i \leq \frac{1}{2}n$,

then π *is forcibly* $1/k$*-tough.*

Proof of Lemma 5.1: Suppose π has a realization G which is not $1/k$-tough. By (ii) and Theorem 1.2, G is connected. So we may assume (by adding edges if necessary) that there exists $X \subseteq V(G)$, with $x \doteq |X| \geq 1$, such that $G = K_x + (K_{a_1} \cup K_{a_2} \cup \cdots \cup K_{a_{kx+1}})$, where $1 \leq a_1 \leq a_2 \leq \cdots \leq a_{kx+1}$.

Set $i \doteq x+k-2+a_{kx}$.

Claim 1. $k \leq i < \frac{1}{2}(n+k-1)$

Proof: The fact that $i \geq k$ follows immediately from the definition of i. Since $kx-x-k+1 = (k-1)(x-1) \geq 0$, we have

$$kx-1 \;\geq\; x+k-2. \tag{3}$$

This leads to

$$\begin{aligned} n \;=\; x + \sum_{j=1}^{kx-1} a_j + a_{kx} + a_{kx+1} \;&\geq\; x+kx-1+2a_{kx} \\ &\geq\; 2x+k-2+2a_{kx} \;=\; 2i-k+2, \end{aligned}$$

which is equivalent to $i < \frac{1}{2}(n+k-1)$. □

Claim 2. $d_i \leq i-k+1$.

Proof: From (3) we get

$$i \;=\; x+k-2+a_{kx} \;\leq\; kx-1+a_{kx} \;\leq\; \sum_{j=1}^{kx} a_j. \tag{4}$$

This gives $d_i \leq x + (a_{kx}-1) \;=\; i-k+1$. □

Claim 3. $d_{n-i+k-1} < n-i$.

Proof: We have $n-i+k-1 = n-x-a_{kx}+1 \leq \sum_{j=1}^{kx+1} a_j$. Thus, using the bound (4) for i,

$$d_{n-i+k-1} \;\leq\; x+a_{kx+1}-1 \;<\; n-\sum_{j=1}^{kx} a_j \;\leq\; n-i.$$ □

Claims 1, 2 and 3 together contradict condition (i), completing the proof of the lemma ■

We can extend Lemma 5.1 to arbitrary $t \le 1$ by letting $k = \lfloor 1/t \rfloor$.

Theorem 5.2. *Let $t \le 1$, $n \ge \lfloor 1/t \rfloor + 2$, and $\pi = (d_1 \le \cdots \le d_n)$ a graphical sequence. If*

(i) $d_i \ge i - \lfloor 1/t \rfloor + 2$ *or* $d_{n-i+\lfloor 1/t \rfloor -1} \ge n - i$, *for* $\lfloor 1/t \rfloor \le i < \frac{1}{2}\big(n + \lfloor 1/t \rfloor - 1\big)$, *and*

(ii) $d_i \ge i$ *or* $d_n \ge n - i$, *for* $1 \le i \le \frac{1}{2}n$,

then π is forcibly t-tough.

Proof: Set $k = \lfloor 1/t \rfloor \ge 1$. If π satisfies conditions (i), (ii) in Theorem 5.2, then π satisfies conditions (i), (ii) in Lemma 5.1, and so is forcibly $1/k$-tough. But $k = \lfloor 1/t \rfloor \le 1/t$ means $1/k \ge t$, and so π is forcibly t-tough. ■

In summary, if $\dfrac{1}{k+1} < t \le \dfrac{1}{k}$ for some integer $k \ge 1$, then Theorem 5.2 declares π forcibly t-tough precisely if Lemma 5.1 declares π forcibly $1/k$-tough.

Acknowledgements.
The authors thank two anonymous referees for comments and suggestions that greatly improved the structure and clarity of the paper. We also thank Michael Yatauro for providing the short argument for Claim 2 in the proof of Theorem 2.1.

References

[1] M. Aigner and E. Triesch. Realizability and uniqueness in graphs. *Discrete Math.* 136 (1994), 3–20.

[2] F. Boesch. The strongest monotone degree condition for n-connectedness of a graph. *J. Comb. Theory Ser. B* 16 (1974), 162–165.

[3] J.A. Bondy. Properties of graphs with constraints on degrees. *Studia Sci. Math. Hungar.* 4 (1969), 473–475.

[4] V. Chvátal. On Hamilton's ideals. *J. Comb. Theory Ser. B* 12 (1972), 163–168.

[5] G.H. Hardy and S. Ramanujan. Asymptotic formulae in combinatory analysis. *Proc. London Math. Soc.* 17 (1918), 75–115.

[6] M. Kriesell. Degree sequences and edge connectivity. Preprint (2007). Available online at `http://www.math.uni-hamburg.de/research/papers/hbm/hbm2007282.pdf`; accessed 23 July 2009.

[7] D. West. *Introduction to Graph Theory (2nd ed.)*, Prentice Hall, Upper Saddle River, New Jersey, 2001.