Toughness and Binding Number

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Abstract

Let $\tau(G)$ and $\mathrm{bind}(G)$ be the toughness and binding number, respectively, of a graph G. Woodall observed in 1973 that $\tau(G) \geqslant \mathrm{bind}(G) - 1$. In this paper we obtain best possible improvements of this inequality except when $(1+\sqrt{5})/2 < \mathrm{bind}(G) < 2$ and $\mathrm{bind}(G)$ has even denominator when expressed in lowest terms.

1 Introduction

We consider only finite undirected simple graphs. Our terminology and notation will be standard, except as indicated. A good reference for any undefined terms or notation is [8]. We mention only that for two graphs G and H with disjoint vertex sets, we will use $G \cup H$ to denote their disjoint union and $G + H$ to denote their join.

Chvátal introduced the notion of the toughness of a graph in [4]. Let $\omega(G)$ denote the number of components of a graph G. For $t \geqslant 0$, we call G *t-tough* if $t \cdot \omega(G - X) \leqslant |X|$ for every $X \subseteq V(G)$ with $\omega(G - X) \geqslant 2$. The *toughness* of G, denoted $\tau(G)$, is the maximum $t \geqslant 0$ for which G is t-tough, so that

$$\tau(G) = \min \left\{ \frac{|X|}{\omega(G-X)} \,\middle|\, X \subset V(G) \text{ and } \omega(G-X) \geqslant 2 \right\}.$$

By convention, $\tau(K_n) := (n-1)$. If G is not complete, we call $X \subseteq V(G)$ a *tough set* of G if $\omega(G - X) \geqslant 2$ and $\tau(G) = |X|/\omega(G - X)$.

In [9], Woodall introduced the notion of the binding number of a graph G. If $S \subseteq V(G)$, let $N(S)$ denote the set of neighbors of S in G, including any vertices of S that have neighbors in S. For $b \geqslant 0$, we call G *b-binding* if $b|S| \leqslant |N(S)|$ for all $S \subseteq V(G)$ with $N(S) \neq V(G)$. The *binding number* of G, denoted $\mathrm{bind}(G)$, is the

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maximum $b \geqslant 0$ such that G is b-binding. Thus,

$$\mathrm{bind}(G) = \min\left\{ \frac{|N(S)|}{|S|} \,\middle|\, \emptyset \neq S \subseteq V(G),\ N(S) \neq V(G) \right\}.$$

We call $S \subseteq V(G)$ a *binding set* of G if $N(S) \neq V(G)$ and $\mathrm{bind}(G) = |N(S)|/|S|$. In particular, $\mathrm{bind}(K_n) = n - 1$.

Toughness and binding number, like vertex-connectivity and edge-connectivity, are both measures of the vulnerability of a graph. Other measures of vulnerability, such as tenacity and integrity, are discussed in [1] and [7]. Vulnerability parameters are of considerable interest in the study of network stability.

An important difference between toughness and binding number is in regard to their computational complexity. It was shown in [2] that deciding if G is t-tough is NP-hard for any rational $t > 0$, and remains so for $t = 1$ even when G is restricted to the class of cubic graphs [3]. By contrast, Cunningham [5] showed that $\mathrm{bind}(G)$ can be determined in polynomial time. This suggests that tight bounds for toughness, in terms of binding number, might be both useful and interesting.

In [9], Woodall proved the following lower bound for $\tau(G)$ in terms of $\mathrm{bind}(G)$.

Theorem 1.1. *Let G be a graph. Then $\tau(G) \geqslant \mathrm{bind}(G) - 1$.*

Woodall noted that the lower bound for $\tau(G)$ in Theorem 1.1 is certainly not best possible, but he made no attempt to improve it. One main goal in the present paper is to obtain best possible strengthenings of Theorem 1.1. We first dispose of a few easy cases in the following theorem, which is proved in Section 2.

Theorem 1.2. *Let G be a graph.*

(*a*) *If $\tau(G) = 0$ then $0 \leqslant \mathrm{bind}(G) \leqslant 1$, and all rational numbers in this interval are possible values for $\mathrm{bind}(G)$.*
(*b*) *If $\mathrm{bind}(G) \leqslant 1$ then $0 \leqslant \tau(G) \leqslant \mathrm{bind}(G)$, and all rational numbers in this interval are possible values for $\tau(G)$.*

In view of Theorem 1.2, we assume in what follows that $\tau(G) > 0$ and $\mathrm{bind}(G) > 1$. In Section 3 we prove the following upper bound for $\mathrm{bind}(G)$ in terms of $\tau(G)$.

Theorem 1.3. *If $\tau(G) \in \mathbb{Z}$, let $c := 2\tau(G)$ and $d := 2$; otherwise let $\tau(G) = c/d$ in lowest terms. Then*

$$\mathrm{bind}(G) \leqslant \max\left\{ \left(\tfrac{d}{2d-1}\right)\tau(G) + 1, \tau(G) \right\} \tag{1}$$

$$= \begin{cases} c/(2d-1) + 1 & \text{if } c \leqslant 2d + 1, \\ c/d & \text{if } c \geqslant 2d + 2. \end{cases} \tag{2}$$

Moreover, this bound is sharp for all possible values of c and d.

Corollary 1.4. *For any graph G,* $\operatorname{bind}(G) \leqslant \max\left\{\frac{2}{3}\tau(G)+1, \tau(G)\right\}$.

Obtaining a best possible lower bound for $\tau(G)$ in terms of $\operatorname{bind}(G)$ turns out to be substantially more difficult. Because of the nature of our results, we do not see any way of getting them from Theorem 1.3 by pure algebra. Of course, one can easily reverse Corollary 1.4:

Corollary 1.5. *For any graph G,* $\tau(G) \geqslant \min\left\{\frac{3}{2}(\operatorname{bind}(G)-1), \operatorname{bind}(G)\right\}$.

While certainly better than Theorem 1.1, Corollary 1.5 is still far from best possible. We will improve it in Section 4, by proving Theorems 1.6 and 1.7 below.

Theorem 1.6. *Suppose* $1 < \operatorname{bind}(G) < 2$*, and let* $\operatorname{bind}(G) - 1 = \alpha/\beta$ *in lowest terms. Then* $\tau(G) \geqslant (\beta/\lceil \frac{\beta}{2} \rceil)(\operatorname{bind}(G)-1) = \alpha/\lceil \frac{\beta}{2} \rceil$ *if either β is odd or* $1 < \operatorname{bind}(G) < \phi$*, where* $\phi := (1+\sqrt{5})/2$.

The upper bound ϕ in Theorem 1.6 is tight, and the appearance of the golden ratio in this way is rather surprising. For $\phi < \operatorname{bind}(G) < 2$ and even β, we show in Section 4 that there are graphs G with $\operatorname{bind}(G)$ arbitrarily close to ϕ and arbitrarily close to 2 for which $\tau(G) < (\beta/\lceil \frac{\beta}{2} \rceil)(\operatorname{bind}(G)-1) = 2(\operatorname{bind}(G)-1)$. In this range we have not proved any result better than Corollary 1.5, although we do not suggest that it is sharp.

Theorem 1.7. *Suppose* $\operatorname{bind}(G) \geqslant 2$*. Then*

$$\tau(G) \geqslant \begin{cases} 3/2 & \text{if } \operatorname{bind}(G) = 2, \\ 2 & \text{if } \operatorname{bind}(G) = 9/4 \text{ or } 2 + 1/(2m-1) \text{ for some } m \geqslant 2, \\ 2 + 1/m & \text{if } \operatorname{bind}(G) = 2 + 2/(2m-1) \text{ for some } m \geqslant 2, \\ \operatorname{bind}(G) & \text{otherwise.} \end{cases}$$

Moreover, these bounds are sharp for every possible value of $\operatorname{bind}(G) \geqslant 2$.

In Theorems 4.4 and 4.5, we describe the forms of all graphs G such that $\tau(G) < \operatorname{bind}(G)$ and $\operatorname{bind}(G) \geqslant 2$.

2 Examples and preliminaries

In this section we first give examples showing that the bounds given in Theorems 1.3, 1.6 and 1.7 are sharp. We then prove Theorem 1.2, and finally we prove a lemma that will be used in the proofs of the remaining theorems.

Example 1. For integers $c \geqslant 1$ and $d \geqslant 2$, let $G := K_c + dK_1$. Then the (unique) tough set is $V(K_c)$, the binding set is $V(dK_1)$, and $\tau(G) = \operatorname{bind}(G) = c/d$.

Example 2. For integers $c \geqslant 1$ and $d \geqslant 2$, let $G := K_c + dK_2$. Then the (unique) tough set is $V(K_c)$, a binding set is $V(dK_2) - \{v\}$ where v is any vertex of dK_2, and $\tau(G) = c/d$ and $\operatorname{bind}(G) = (c + 2d - 1)/(2d - 1) = c/(2d - 1) + 1$.

Examples 1 and 2 show that the bound in Theorem 1.3 is sharp. The bound in Theorem 1.6 is sharp when β is odd, by Example 2 with $c := \alpha$ and $d := \lceil \frac{\beta}{2} \rceil \geqslant 2$ (clearly $\beta > 1$), so that $\beta = 2d - 1$. To see that it is sharp when β is even, let $G := K_{2\alpha} + (K_3 \cup (\beta - 1)K_2)$ and let $S := V(K_3 \cup (\beta - 1)K_2) - \{v\}$, where $v \in V((\beta - 1)K_2)$. Then

$$\operatorname{bind}(G) = \frac{|N(S)|}{|S|} = \frac{2\alpha + 2\beta}{2\beta} = \frac{\alpha}{\beta} + 1 \quad \text{and} \quad \tau(G) = \frac{2\alpha}{\beta} = \frac{\alpha}{\lceil \frac{\beta}{2} \rceil}.$$

Together with Example 1, the following examples show that the bounds in Theorem 1.7 are sharp. In the second of these, a binding set is of the form $V(2K_3) - \{u, v\}$, where u, v are two adjacent vertices of $V(2K_3)$; the other examples are all special cases of Example 2.

$$\begin{array}{lll}
G = K_3 + 2K_2, & \tau(G) = 3/2, & \operatorname{bind}(G) = 2; \\
G = K_4 + 2K_3, & \tau(G) = 2, & \operatorname{bind}(G) = 9/4; \\
G = K_{2m} + mK_2 \ (m \geqslant 2), & \tau(G) = 2, & \operatorname{bind}(G) = 2 + 1/(2m - 1); \\
G = K_{2m+1} + mK_2 \ (m \geqslant 2), & \tau(G) = 2 + 1/m, & \operatorname{bind}(G) = 2 + 2/(2m - 1).
\end{array}$$

We now prove Theorem 1.2. We need the following lemma.

Lemma 2.1. *Let a, b, c, d be integers such that b, d are positive and $0 \leqslant a/b \leqslant c/d \leqslant 1$. Then there is a graph G such that $\tau(G) = a/b$ and $\operatorname{bind}(G) = c/d$.*

Proof. If $a = c = 0$, take $G := K_1$. If $a = 0 < c$, take $G := K_2 \cup K_{c,d}$. If $a > 0$, take $G := K_x + (yK_1 \cup zK_2)$, where $x := 2ac$, $y := 2ad$ and $z := 2bc - 2ad$. Since the possible binding sets are of the form $V(yK_1 \cup H)$ where $H \subseteq zK_2$, and since $x \leqslant y$, a binding set is $V(yK_1)$, and $\operatorname{bind}(G) = x/y = c/d$. The (unique) tough set is $V(K_x)$, and $\tau(G) = x/(y + z) = a/b$. $\square$

Proof of Theorem 1.2. If $\tau(G) = 0$ then G is disconnected and so $\operatorname{bind}(G) \leqslant 1$. It was proved in [6] that if $\operatorname{bind}(G) \leqslant 1$ then $\tau(G) \leqslant \operatorname{bind}(G)$. The result now follows from Lemma 2.1. $\square$

We will make extensive use of the following lemma.

Lemma 2.2. *Let G be a graph such that $\operatorname{bind}(G) > 1$ and G is not complete. Let X be a tough set of G and define $x := |X|$ and $\omega := \omega(G - X) \geqslant 2$, so that $\tau(G) = x/\omega$. Let Y_0 be the vertex-set of a smallest component of $G - X$, and let $y_0 := |Y_0|$.*
(a) If $y_0 = 1$, let $j < \omega$ be the number of nontrivial components of $G - X$. Then

$$\operatorname{bind}(G) \leqslant \frac{x + 2j}{\omega + j} = \frac{\tau(G)\omega + 2j}{\omega + j}. \tag{3}$$

(b) If $y_0 = 2$ then $(\mathrm{bind}(G) - 1)(|V(G - X)| - 1) \leqslant x = \tau(G)\omega$.

(c) If $y_0 \geqslant 3$ then $(\mathrm{bind}(G) - 1)(|V(G - X)| - y_0) \leqslant x = \tau(G)\omega$.

Proof. To prove (a), let $S := V(G - X)$ and $s := |S|$. Then

$$\mathrm{bind}(G) \leqslant \frac{|N(S)|}{|S|} \leqslant \frac{x + s - (\omega - j)}{s}. \tag{4}$$

The hypothesis that $\mathrm{bind}(G) > 1$ implies that $x > \omega - j$. Thus the RHS of (4) is largest when s is as small as possible, that is, $s = \omega + j$. Substituting this value in (4) gives the result.

We prove (b) and (c) together. If $y_0 = 2$ let $S := V(G) - (X \cup \{v\})$, where $v \in Y_0$, so that $|S| = |V(G - X)| - 1$. If $y_0 \geqslant 3$ let $S := V(G) - (X \cup Y_0)$, so that $|S| = |V(G - X)| - y_0$. In either case, $|N(S)| \leqslant x + |S|$, and so $\mathrm{bind}(G) - 1 \leqslant |N(S)|/|S| - 1 \leqslant x/|S|$. This completes the proof of Lemma 2.2. $\square$

3 An upper bound for binding number in terms of toughness

In this section we prove Theorem 1.3. To see that (1) and (2) are equal, note that $2d + 1 < \frac{d(2d-1)}{d-1} \leqslant 2d + 2$ since $d \geqslant 2$, and so

$$\frac{c}{d} \geqslant \frac{c}{2d-1} + 1 \iff c \geqslant \frac{d(2d-1)}{d-1} \iff c \geqslant 2d + 2$$

since c is an integer.

It was shown in Section 2 that the bound in Theorem 1.3 is sharp for all possible values of c and d. It remains to prove the bound. It is easy to see that the result holds if $\mathrm{bind}(G) \leqslant 1$ or if $G = K_n$, when $\mathrm{bind}(G) = n - 1 = \tau(G)$. So assume that $\mathrm{bind}(G) > 1$ and G is not complete. As in Lemma 2.2, let X be a tough set of G, define $x := |X|$ and $\omega := \omega(G - X) \geqslant 2$, so that $\tau(G) = x/\omega$, and let y_0 be the order of a smallest component of $G - X$.

Case 1: $y_0 = 1$. Let $j < \omega$ be the number of nontrivial components of $G - X$. The RHS of (3) is largest when $j = \omega - 1$ if $\tau(G) \leqslant 2$ and when $j = 0$ if $\tau(G) \geqslant 2$, and $\frac{(\tau(G)+2)\omega}{2\omega} \leqslant \frac{2}{1}$ if $\tau(G) \leqslant 2$, so that

$$\mathrm{bind}(G) \leqslant \begin{cases} \frac{(\tau(G)+2)\omega - 2}{2\omega - 1} \leqslant \frac{\tau(G)+2}{2} & \text{if } \tau(G) \leqslant 2, \\ \frac{\tau(G)\omega}{\omega} = \tau(G) & \text{if } \tau(G) \geqslant 2. \end{cases}$$

But $\frac{\tau(G)+2}{2} < \left(\frac{d}{2d-1}\right) \tau(G) + 1$, and so (1) holds.

Case 2: $y_0 \geqslant 2$. Note first that if $\tau(G) \in \mathbb{Z}$ then $\omega \geqslant d = 2$, and if $\tau(G) \notin \mathbb{Z}$ then $x/\omega = \tau(G) = c/d$ in lowest terms, so that $\omega \geqslant d$. Next note that if $y_0 = 2$ then

$|V(G) - X| - 1 \geqslant 2\omega - 1$, and if $y_0 \geqslant 3$ then $|V(G) - X| - y_0 \geqslant 3(\omega - 1) \geqslant 2\omega - 1$. It now follows from parts (b) and (c) of Lemma 2.2 that

$$\mathrm{bind}(G) \leqslant \left(\frac{\omega}{2\omega - 1}\right) \tau(G) + 1 \leqslant \left(\frac{d}{2d - 1}\right) \tau(G) + 1.$$

This completes the proof of (1) and hence of Theorem 1.3. $\square$

The fact that $\frac{d}{2d-1} \leqslant \frac{2}{3}$ for $d \geqslant 2$ yields Corollary 1.4, and Corollary 1.5 immediately follows.

4 A lower bound for toughness in terms of binding number

In this section we prove Theorems 1.6 and 1.7. We need the following result, which follows from the fact that $\mathrm{bind}(G) \leqslant |N(S_v)|/|S_v| \leqslant (n-1)/(n-d(v))$ for every vertex v of G, where $S_v := V(G) - N(v)$.

Theorem 4.1. [9, Corollary 7.1] *Let G be a graph on n vertices. If $\mathrm{bind}(G) \geqslant b > 0$, then*

$$\delta(G) \geqslant \frac{(b-1)n + 1}{b}. \tag{5}$$

We use this result in the proof of the following lemma.

Lemma 4.2. *Let G be a graph of order n containing a set X of x vertices such that $G - X$ has $\omega = \omega(G - X)$ components $H_1, \ldots, H_\omega$, all with the same order k. Suppose that either*

(i) $\mathrm{bind}(G) > \dfrac{n-1}{n - (x + k - 2)}$,

or

(ii) $k \geqslant 2$, $\omega \geqslant 2$, $\mathrm{bind}(G) \geqslant 2$, $x \leqslant 2\omega + 1$, *and if* $\mathrm{bind}(G) = 2$ *then* $x \leqslant 2\omega$.

Then $G[X]$, the subgraph of G induced by X, has order x and minimum degree at least $x + k - 1 - k\omega$, and $G = G[X] + \omega K_k$.

Proof. The inequality for x in (ii) implies that

$$x \leqslant 2\omega + 1 + (k-2)(\omega - 2) = k\omega - 2(k-2) + 1,$$

which is the same as saying that $-1 \leqslant n - 2(x + k - 2)$, since clearly $n = k\omega + x$. Thus $n - 1 \leqslant 2(n - (x + k - 2))$, so that the RHS of (i) is at most 2, and is less than 2 if the inequality given for x is strict. Thus (ii) implies (i). So assume that (i) holds, and let b denote the RHS of (i). Then

$$b(x + k - 2) = bn - (n - 1) = (b - 1)n + 1,$$

and the RHS of (5) is equal to $x+k-2$. Since $\mathrm{bind}(G) > b$ and the RHS of (5) is an increasing function of b, it follows from Theorem 4.1 that $\delta(G) > x+k-2$, so that $\delta(G) \geqslant x+k-1$. Thus every vertex of H_i ($1 \leqslant i \leqslant \omega$) is adjacent to all the other $x+k-1$ vertices in $X \cup V(H_i)$, which means that $G = G[X] + \omega K_k$. And every vertex of X is adjacent to at least $x+k-1-k\omega$ other vertices of X, which completes the proof. $\square$

4.1 $1 < \mathrm{bind}(G) < 2$

In this subsection we prove Theorem 1.6 and give some examples of graphs with even β such that $\phi < \mathrm{bind}(G) < 2$ and $\tau(G) < 2(\mathrm{bind}(G)-1)$.

Proof of Theorem 1.6. Let $b := \mathrm{bind}(G)$, so that $b-1 = \alpha/\beta$ in lowest terms and $0 < \alpha/\beta < 1$. Suppose that $\tau(G) < (\beta/\lceil \frac{\beta}{2} \rceil)(b-1) = \alpha/\lceil \frac{\beta}{2} \rceil$. Since $1 < \mathrm{bind}(G) < 2$, G is connected but not complete. Let X be a tough set of G, and let $x := |X| > 0$ and $\omega := \omega(G-X)$. Note that $x/\omega = \tau(G) < \alpha/\lceil \frac{\beta}{2} \rceil$, so that

$$x < \frac{\alpha\omega}{\lceil \frac{\beta}{2} \rceil} \leqslant \frac{2\alpha\omega}{\beta} < 2\omega. \tag{6}$$

Also,

$$b-1 = \frac{\alpha}{\beta} > \frac{x}{2\omega}. \tag{7}$$

Finally, let Y_0 (Y_1) be the vertex-set of a smallest (largest) component of $G-X$, and let $y_i := |Y_i|$ ($i = 0,1$). We consider several cases.

Case 1: $y_0 = 1$. Let $j < \omega$ be the number of nontrivial components of $G-X$. Then

$$\frac{\alpha+\beta}{\beta} = \frac{\alpha}{\beta} + 1 = b = \mathrm{bind}(G) \leqslant \frac{x+2j}{\omega+j}$$

by Lemma 2.2(a). After rearranging, we get

$$(\beta-\alpha)j \geqslant (\alpha+\beta)\omega - \beta x > (\beta-\alpha)\omega,$$

where the last inequality holds since $\beta x < 2\alpha\omega$ by (6). Since $\alpha < \beta$, it follows that $j > \omega$, a contradiction.

Case 2: $y_0 \geqslant 3$. Let $s := |V(G-X)| - y_0 \geqslant 3(\omega-1)$. By Lemma 2.2(c) and (6),

$$\frac{\alpha}{\beta} = \mathrm{bind}(G) - 1 \leqslant \frac{x}{s} < \frac{2\alpha\omega}{\beta s}$$

so that $s < 2\omega$. Thus $3\omega - 3 < 2\omega$, which implies $\omega = 2$; and $y_1 = s < 4$, so that $y_0 = y_1 = 3$. By (7) with $\omega = 2$,

$$b = \mathrm{bind}(G) > \frac{x}{4} + 1 > \frac{x}{5} + 1 = \frac{(x+6)-1}{(x+6)-(x+1)},$$

and Lemma 4.2 with $n = x + 6$ and $k = 3$ implies $G = G[X] + 2K_3$.

By (6), $x < 2\omega = 4$, so that $1 \leqslant x \leqslant 3$, and $\tau(G) = x/\omega = x/2$. Since $\mathrm{bind}(G) < 2$, Y_0 is a binding set of G, which implies that $\mathrm{bind}(G) = |N(Y_0)|/|Y_0| = 1 + x/3 < 2$ and so $x < 3$. Thus $1 \leqslant x \leqslant 2$, so that $\alpha = x$ and $\beta = 3$, and $\tau(G) = x/2 = \alpha/\left\lceil \frac{\beta}{2} \right\rceil$, a contradiction.

Case 3: $y_0 = 2$. Let $s := |V(G - X)| - 1$. Then Lemma 2.2(b) implies $s < 2\omega$ exactly as Lemma 2.2(c) did in Case 2. Hence every component of $G - X$ has order 2, and $s = 2\omega - 1$. By (7),

$$b = \mathrm{bind}(G) > \frac{x}{2\omega} + 1 = \frac{x + 2\omega}{2\omega} > \frac{(x + 2\omega) - 1}{(x + 2\omega) - x}.$$

Now Lemma 4.2 with $n = x + 2\omega$ and $k = 2$ implies that $G = G[X] + \omega K_2$.

By Lemma 2.2(b), $(b - 1)s \leqslant x$ and so

$$\frac{\alpha}{\beta} = b - 1 \leqslant \frac{x}{s} = \frac{x}{2\omega - 1}. \tag{8}$$

Suppose equality holds in (8); then β is an odd integer with $\beta \leqslant 2\omega - 1$, and

$$\tau(G) = \frac{x}{\omega} = \frac{s}{\omega} \cdot \frac{x}{s} = \frac{2\omega - 1}{\omega}(b - 1) \geqslant \frac{\beta}{\left\lceil \frac{\beta}{2} \right\rceil}(b - 1),$$

a contradiction. This shows that

$$b = \mathrm{bind}(G) < \frac{x}{2\omega - 1} + 1. \tag{9}$$

Let S be a binding set of G. Note that S cannot intersect both $G[X]$ and ωK_2, otherwise $N(S) = V(G)$. If $S \subseteq V(\omega K_2)$ then $S = V(\omega K_2) - \{v\}$, where $v \in V(\omega K_2)$, and $b = \mathrm{bind}(G) = |N(S)|/|S| = (x + s)/s$; thus equality holds in (8), a contradiction. Hence $S \subseteq V(G[X])$, and so $|S| \leqslant x$.

Suppose first that β is odd. By (9),

$$\frac{2\omega - 1}{\omega}(b - 1) < \frac{x}{\omega} = \tau(G) < \frac{\beta}{\left\lceil \frac{\beta}{2} \right\rceil}(b - 1),$$

and so $\beta > 2\omega - 1$. Thus $\beta \geqslant 2\omega + 1$. Since $|N(S)|/|S| = \mathrm{bind}(G) = \alpha/\beta + 1 = (\alpha + \beta)/\beta$ (lowest terms), we have $|S| \geqslant \beta$. However, this implies that $x \geqslant |S| \geqslant \beta \geqslant 2\omega + 1$, contradicting (6). We conclude that a counterexample G cannot exist if β is odd.

Finally, suppose $1 < \mathrm{bind}(G) < \phi$. Since we are assuming that $x/\omega = \tau(G) < 2(\mathrm{bind}(G) - 1) = 2(b - 1)$, it follows that

$$b\tau(G) < 2b(b - 1) < 2\phi(\phi - 1) = 2.$$

Thus

$$\operatorname{bind}(G) = b < \frac{2}{\tau(G)} = \frac{2\omega}{x}. \tag{10}$$

Now, $\operatorname{bind}(G) \geqslant \operatorname{bind}(xK_1 + \omega K_2)$. It is easily verified that $\operatorname{bind}(xK_1 + \omega K_2)$ is either $(x + 2\omega - 1)/(2\omega - 1)$ or $2\omega/x$. The former contradicts (9) and the latter contradicts (10). We conclude that a counterexample G cannot exist if $1 < \operatorname{bind}(G) < \phi$. $\square$

We now show that the upper bound ϕ in Theorem 1.6 is best possible when β is even. Consider the Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for $n \geqslant 2$. It is well known that F_{6n} is even and $\frac{F_{6n+1}}{F_{6n}}$ is in lowest terms. Define $G_n := F_{6n+1}K_2 + \overline{K_{2F_{6n}}}$, for $n \geqslant 1$.

Claim.

$$\operatorname{bind}(G_n) = F_{6n+1}/F_{6n} \searrow \phi \text{ as } n \to \infty, \tag{11}$$

$$\tau(G_n) = 2F_{6n}/F_{6n+1}, \tag{12}$$

$$\tau(G_n) < 2(\operatorname{bind}(G_n) - 1). \tag{13}$$

Proof. For (11), let $k := 6n$, so that $G = F_{k+1}K_2 + \overline{K_{2F_k}}$. The only possible binding sets for G are $V(\overline{K_{2F_k}})$ and $V(F_{k+1}K_2) - \{v\}$, where $v \in V(F_{k+1}K_2)$. Comparing the two resulting candidates for $\operatorname{bind}(G)$, it suffices to show that

$$\frac{F_{k+1}}{F_k} \leqslant \frac{2F_k + 2F_{k+1} - 1}{2F_{k+1} - 1},$$

or

$$2F_{k+1}^2 - F_{k+1} \leqslant F_k(2F_k + 2F_{k+1} - 1) = 2F_kF_{k+2} - F_k,$$

or

$$2(-1)^k = 2(F_{k+1}^2 - F_kF_{k+2}) \leqslant F_{k+1} - F_k = F_{k-1},$$

where the first equality is Cassini's identity. However, $F_{k-1} \geqslant 2(-1)^k$ if $k \geqslant 3$, proving the equality in (11). It is well known that $\frac{F_{6n+1}}{F_{6n}} \searrow \phi$.

For (12), we note that the only two possible values for $\tau(G_n)$ are $\frac{2F_{6n}}{F_{6n+1}}$ and $\frac{F_{6n+1}}{F_{6n}}$, and that $\frac{2F_{6n}}{F_{6n+1}} < \frac{F_{6n+1}}{F_{6n}}$, since $\frac{F_{6n+1}}{F_{6n}} \searrow \phi > \sqrt{2}$.

Using (11) and (12), we see that (13) is equivalent to

$$2\frac{F_{6n}}{F_{6n+1}} < 2\left(\frac{F_{6n+1}}{F_{6n}} - 1\right),$$

or

$$\frac{F_{6n}}{F_{6n+1}} < \frac{F_{6n+1} - F_{6n}}{F_{6n}} = \frac{F_{6n-1}}{F_{6n}},$$

or $F_{6n}^2 - F_{6n-1}F_{6n+1} < 0$, which is true since $F_{6n}^2 - F_{6n-1}F_{6n+1} = (-1)^{6n-1} = -1$. $\square$

The largest binding number in the family of graphs above is $F_7/F_6 = 13/8$. We now demonstrate that the inequality $\tau(G) \geqslant 2(\mathrm{bind}(G) - 1)$ of Theorem 1.6 fails for a sequence of larger binding numbers approaching 2 from below. Consider the graphs $G_t := aK_2 + (K_b + \overline{K_c})$, where $a = 8+3t$, $b = 1+2t$ and $c = 10+4t$, for $t \geqslant 0$. The possible binding sets of G_t are $V(\overline{K_c})$ and $V(aK_2) - \{v\}$, for $v \in aK_2$. Similarly, the possible tough sets of G_t are $V(K_b + \overline{K_c})$ and $V(aK_2 + K_b)$. By comparing the appropriate ratios in each case, we find that the binding set is $V(\overline{K_c})$ and the tough set is $V(K_b + \overline{K_c})$, so that

$$\mathrm{bind}(G_t) = \frac{2a+b}{c} = \frac{17+8t}{10+4t} = 1 + \frac{7+4t}{10+4t}$$

and

$$\tau(G_t) = \frac{b+c}{a} = \frac{11+6t}{8+3t}.$$

The denominator β of $\frac{7+4t}{10+4t}$ in lowest terms is even. Moreover,

$$\tau(G_t) = \frac{11+6t}{8+3t} < \frac{7+4t}{5+2t} = 2(\mathrm{bind}(G_t) - 1),$$

and thus the inequality in Theorem 1.6 fails for G_t, for every $t \geqslant 0$. Finally, $\mathrm{bind}(G_t) \nearrow 2$ as $t \to \infty$, providing the desired examples.

4.2 $\mathrm{bind}(G) \geqslant 2$

In this subsection we prove Theorem 1.7. We showed in Section 2 that the bounds given in Theorem 1.7 are all sharp; it remains to prove the bounds. They follow from the more general results in Theorems 4.4 and 4.5 below, for the cases $\mathrm{bind}(G) = 2$ and $\mathrm{bind}(G) > 2$, respectively. We first need a lemma.

Lemma 4.3. *Let G be a graph and X a tough set of G. If $\mathrm{bind}(G) \geqslant 2$ and $\tau(G) < \mathrm{bind}(G)$, then the components of $G - X$ all have the same order $k \geqslant 2$.*

Proof. Since $\tau(G) \neq \mathrm{bind}(G)$, G is not complete. Let X be a tough set of G. Let $b := \mathrm{bind}(G)$, $x := |X|$, and $\omega := \omega(G - X)$, so that $x < b\omega$. Let y_0 (y_1) be the order of a smallest (largest) component of $G - X$.

If $y_0 = 1$, Lemma 2.2(a) gives $b \leqslant (x + 2j)/(\omega + j) < (b\omega + 2j)/(\omega + j)$, where j is the number of nontrivial components in $G - X$. This rearranges to $j(b - 2) < 0$, which is impossible since $j \geqslant 0$ and $b \geqslant 2$.

Thus $y_0 \geqslant 2$. We wish to prove that $y_1 = y_0$. If $y_1 > y_0 = 2$ then $|V(G-X)| - 1 \geqslant 2\omega$, and Lemma 2.2(b) implies that $(b - 1)(2\omega) \leqslant x < b\omega$. This implies $b < 2$, a contradiction. If $y_1 > y_0 \geqslant 3$, then $|V(G-X)| - y_0 \geqslant (\omega-1)y_0+1$, and Lemma 2.2(c) implies that $(b - 1)((\omega - 1)y_0 + 1) \leqslant x < b\omega$. This rearranges to

$$(b - 1)(\omega - 1)(y_0 - 2) + (b - 2)(\omega - 1) < 1, \tag{14}$$

which is impossible since $b \geqslant 2$, $\omega \geqslant 2$ and $y_0 \geqslant 3$, so that the first term of (14) is at least 1 and the second term is nonnegative. We conclude that $y_1 = y_0$ and all components of $G - X$ have the same order. $\square$

If $\text{bind}(G) = 2$ then $\tau(G) \geqslant 3/2$ by Corollary 1.5, and this is all that is claimed in Theorem 1.7. The following theorem gives more information. Note that the set of graphs $G = G_{2m-1} + mK_2$ in this theorem includes some for which $\tau(G) \geqslant \text{bind}(G)$.

Theorem 4.4. *Let G be a graph such that* $\text{bind}(G) = 2$*. Then $\tau(G) \geqslant \text{bind}(G) = 2$ unless $\tau(G) = 3/2$ and $G = G_3 + 2K_3$ for some graph G_3 of order 3, or $\tau(G) = 2 - 1/m$ and $G = G_{2m-1} + mK_2$ for some graph G_{2m-1} of order $2m - 1$, where $m \geqslant 2$.*

Proof. Assume that $\tau(G) < \text{bind}(G) = 2$. Since $\tau(G) \neq \text{bind}(G)$, G is not complete. Let X be a tough set of G. Define $x := |X|$ and $\omega := \omega(G - X)$, so that $x < 2\omega$. By Lemma 4.3, every component of $G - X$ has the same order $k \geqslant 2$. It follows from Lemma 4.2 that $G = G[X] + \omega K_k$.

If $k = 2$ then $|V(G - X)| - 1 = 2\omega - 1$, and Lemma 2.2(b) with $\text{bind}(G) = 2$ implies that $2\omega - 1 \leqslant x < 2\omega$. Thus $x = 2\omega - 1$, $G = G_{2\omega-1} + \omega K_2$, and $\tau(G) = x/\omega = 2 - 1/\omega$.

So assume that $k \geqslant 3$. Then $|V(G - X)| - k = k(\omega - 1)$, and Lemma 2.2(c) implies that $3(\omega - 1) \leqslant k(\omega - 1) \leqslant x < 2\omega$. Thus $\omega = 2$, $k < 4$ and $3 \leqslant x < 4$. So $k = x = 3$ and $G = G_3 + 2K_3$, and $\tau(G) = x/\omega = 3/2$. $\square$

The following theorem completes the proof of Theorem 1.7 when $\text{bind}(G) > 2$.

Theorem 4.5. *Let G be a graph such that* $\text{bind}(G) = b > 2$*. Then $\tau(G) \geqslant b$ except in the following three cases.*

(i) $b = 9/4$. In this case the exceptional graphs are precisely the graphs of the form $G = H + 2K_3$, where H is a 4-vertex graph with at least two edges. Here $\tau(G) = 2$.

(ii) $b = (4m - 1)/(2m - 1)$ for some $m \geqslant 2$. In this case the exceptional graphs are all of the form $G = H + mK_2$ where $|V(H)| = 2m$, $\delta(H) \geqslant 1$, $\text{bind}(G) = 2 + 1/(2m - 1)$ *and $\tau(G) = 2$.*

(iii) $b = 4m/(2m - 1)$ for some $m \geqslant 2$. In this case the exceptional graphs are all of the form $G = H + mK_2$ where $|V(H)| = 2m + 1$, $\delta(H) \geqslant 2$, $\text{bind}(G) = 2 + 2/(2m - 1)$ *and $\tau(G) = 2 + 1/m$.*

Proof. Let $\text{bind}(G) = b > 2$, and assume that $\tau(G) < b$. Then G is not complete. Let X be a tough set of G. Define $x := |X|$ and $\omega := \omega(G - X)$, so that $x = \tau(G)\omega < b\omega$. By Lemma 4.3, every component of $G - X$ has the same order $k \geqslant 2$.

We wish to prove that G has one of the forms described in parts (i)–(iii) of the theorem. We consider two cases; forms (i) and (ii) arise in Case 1, and form (iii) in Case 2.

Case 1: $\tau(G) \leqslant 2$. Then $x \leqslant 2\omega$, and it follows from Lemma 4.2 that $G = G[X] + \omega K_k$.

Case 1a: $k \geqslant 3$. Then $|V(G-X)| - k = k(\omega - 1)$. By Lemma 2.2(c), and since $b > 2$,

$$3(\omega - 1) < (b-1)k(\omega - 1) \leqslant x \leqslant 2\omega,$$

so that $\omega = 2$, $k = 3$, $x = 4$, and $\tau(G) = x/\omega = 2$. Thus $G = H + 2K_3$, where $H = G[X]$ is a graph of order 4. If $|E(H)| \leqslant 1$, then $2 < b = \mathrm{bind}(G) \leqslant \mathrm{bind}((K_2 \cup 2K_1) + 2K_3) = 2$, a contradiction. But if H is any 4-vertex graph with at least two edges then it is not difficult to see that $b = \mathrm{bind}(G) = 9/4$ and $\tau(G) = 2$, as in Theorem 4.5(i).

Case 1b: $k = 2$. Then $|V(G-X)| - 1 = 2\omega - 1$. By Lemma 2.2(b), and since $b > 2$,

$$2\omega - 1 < (b-1)(2\omega - 1) \leqslant x \leqslant 2\omega.$$

It follows that $x = 2\omega$, so that $\tau(G) = 2$, and $b \leqslant 1 + 2\omega/(2\omega - 1) = 2 + 1/(2\omega - 1)$. Now let S be a binding set of G. Since $b > 2$, $|N(S)| = b|S| \geqslant 2|S| + 1$, and so

$$\frac{1}{|S|} \leqslant \frac{|N(S)| - 2|S|}{|S|} = b - 2 \leqslant \frac{1}{2\omega - 1},$$

which implies that $|S| \geqslant 2\omega - 1$. However, $2|S| < b|S| = |N(S)| < |V(G)| = 4\omega$, and so $|S| \leqslant 2\omega - 1$. Thus $|S| = 2\omega - 1$ and $|N(S)| = 4\omega - 1$, so that $b = (4\omega - 1)/(2\omega - 1)$. It follows from Lemma 4.2 that $G[X]$ has minimum degree at least $x + k - 1 - k\omega = 1$, so that G has the form described in Theorem 4.5(ii).

Case 2: $\tau(G) > 2$. Let $S_0 := V(G - X - Y) \cup \{v\}$, where Y is the vertex-set of a component of $G - X$ and $v \in Y$. Then $|S_0| = k(\omega - 1) + 1$ and $|N(S_0)| \leqslant x + k\omega - 1$, and so

$$\frac{x}{\omega} = \tau(G) < b \leqslant \frac{|N(S_0)|}{|S_0|} \leqslant \frac{x + k\omega - 1}{k(\omega - 1) + 1}, \tag{15}$$

which gives $x(k\omega - k - \omega + 1) < k\omega^2 - \omega$. But $x/\omega = \tau(G) > 2$, and so $x = 2\omega + 1 + \epsilon$, where ϵ is a nonnegative integer. Substituting this in (15) and rearranging, we obtain

$$(k-2)(\omega^2 - \omega - 1) + \epsilon(k-1)(\omega - 1) < 1.$$

This is impossible if $k \geqslant 3$, since $\omega \geqslant 2$ and so $\omega^2 - \omega - 1 \geqslant 1$. Thus $k = 2$ and $\epsilon = 0$, meaning that $x = 2\omega + 1$. Now Lemma 4.2 implies that $G = G[X] + \omega K_2$, where $G[X]$ has order $2\omega + 1$ and minimum degree at least $x + k - 1 - k\omega = 2$. Thus

$$b = \mathrm{bind}(G) \leqslant \mathrm{bind}(K_{2\omega+1} + \omega K_2) = \frac{4\omega}{2\omega - 1} = 2 + \frac{2}{2\omega - 1}. \tag{16}$$

Now, $\tau(G) = x/\omega = (2\omega + 1)/\omega$, and this fraction is clearly in lowest terms. By Theorem 1.3,

$$b \leqslant \left(\frac{\omega}{2\omega - 1}\right)\tau(G) + 1 < \left(\frac{\omega}{2\omega - 1}\right) b + 1.$$

This gives

$$2 + \frac{1}{\omega} = \tau(G) < b < \frac{2\omega - 1}{\omega - 1} = 2 + \frac{1}{\omega - 1}. \tag{17}$$

Let S be a binding set of G. Then $|N(S)| \leqslant |V(G)| - 1 = 4\omega$, and so by (17)

$$\frac{2\omega + 1}{\omega} < b = \frac{|N(S)|}{|S|} \leqslant \frac{4\omega}{|S|},$$

which implies that $|S| < 4\omega^2/(2\omega+1) = 2\omega - 1 + 1/(2\omega+1)$, and so $|S| \leqslant 2\omega - 1$. But also $|N(S)| \geqslant 2|S| + 2$, since $b = |N(S)|/|S| > 2$, and if $|N(S)| = 2|S| + 1$ then $b = 2 + 1/|S|$, which contradicts (17). Thus $b \geqslant 2 + 2/|S|$, and (16) implies that $|S| \geqslant 2\omega - 1$. Thus $|S| = 2\omega - 1$ and $|N(S)| = 2|S| + 2 = 4\omega$, so that $\operatorname{bind}(G) = b = 4\omega/(2\omega - 1) = 2 + 2/(2\omega - 1)$. Hence G has the form described in Theorem 4.5(iii).

This completes the proof of Theorem 4.5. $\square$

Notice that the largest exceptional value of b in Theorem 4.5 is $8/3$, obtained by putting $m = 2$ in (iii), and the largest value of $\operatorname{bind}(G) - \tau(G)$ for the exceptional graphs is $1/3$, obtained by putting $m = 2$ in (ii). This implies the following two corollaries.

Corollary 4.6. *If* $\operatorname{bind}(G) > \frac{8}{3}$*, then* $\tau(G) \geqslant \operatorname{bind}(G)$.

Corollary 4.7. *If* $\operatorname{bind}(G) > 2$*, then* $\tau(G) \geqslant \operatorname{bind}(G) - \frac{1}{3}$.

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